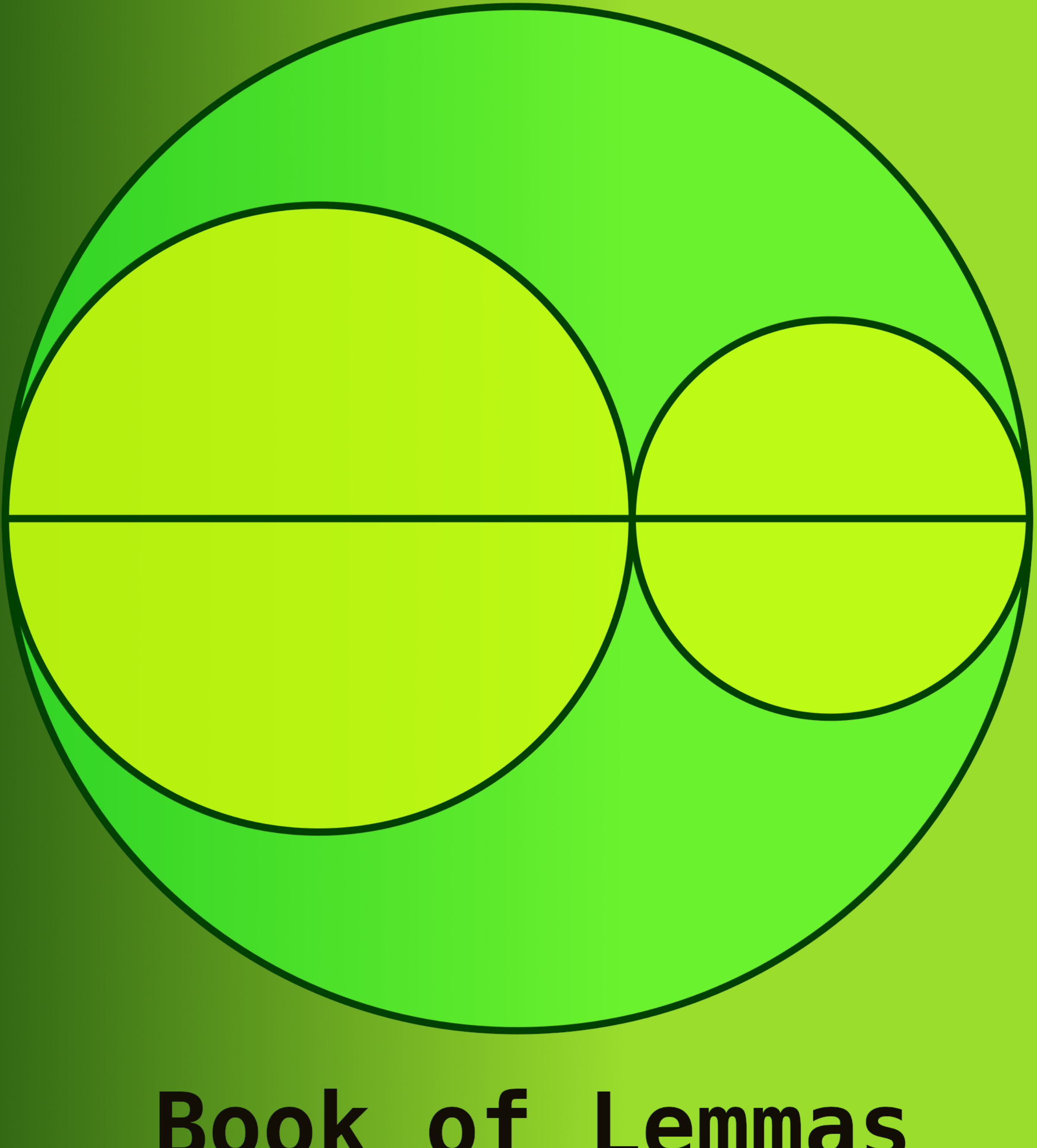
Archimedes



Book of Lemmas

BOOK OF LEMMAS

by
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Prologue

The Book of Lemmas is a treatise of 15 theorems related to the circle. It was first introduced in Arabic by Thebit Ben-Kora who attributed the work to Archimedes. In 1661, the Arabic manuscript was translated into Latin by Abraham Ecchellensis and edited by Giovanni A. Borelli. The Latin version was published under the name Liber Assumptorum. Subsequently the treatise has been translated and in other languages.

The ancient text I used is the Liber assumptorum (grc) which is I found in a collection of thirteen treatises of Archimedes in Perseus Digital Library. This text fully agrees with the text in the publication of Archimedes by Charles Mugler and there are only a few differences with the reconstruction of the ancient text into Sicilian Doric dialect under Evangelos Stamatis. These differences have to do mainly with the translation of a limited number of words and some arithmetic adjectives in the fifteenth lemma. During transportation wherever I considered it more accurate I used the translation be E.S. Stamatis.

The English text can be viewed in parallel with the ancient Greek text so that a reader (who knows both languages) can easily compare the translation achieved. However the criterion for translating the Greek text into English was not to make a rigorous translation. Instead I used a more free translation and modern mathematical symbolism in order to exhibit more clearly the logic of these lemmas and make them easily understood even from a college student.

I would be honored if with this work I contribute at minimum to what Geometry shows us.

Athens, July 2018
Nikolaos L. Kechris

α′.

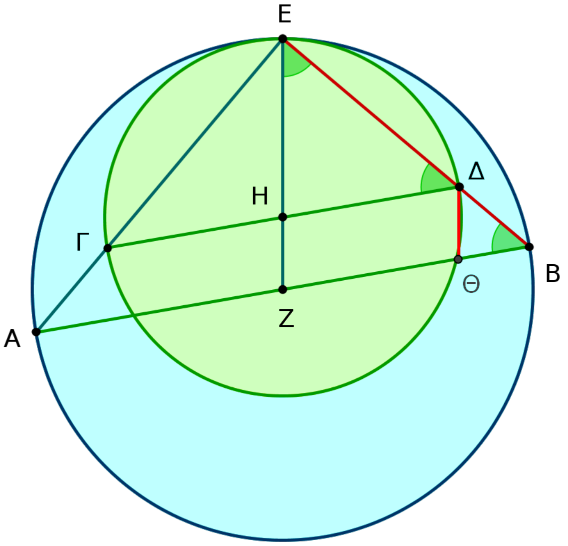
Εἴ κα ᾖ δύο κύκλοι ἐπιψαύοντες ἀλλάλων ἐντός, διάμετροι δὲ αὐτῶν παράλληλοι, ἐπιζευχθεῖσαι αἱ ἀπὸ τοῦ σαμείου ἁφῆς καὶ τῶν περάτων τῶν διαμέτρων δύο εὐθεῖαι ἐσσοῦνται ἀλλάλαις ἐπ’ εὐθείας.

Ἔστωσαν δύο κύκλοι, ὧν κέντρα τὰ **Ζ**, **Η**, ἐπιψαύοντες ἀλλάλων κατὰ τὸ **Ε** σαμεῖον, διάμετρος δὲ ἁ **ΑΒ** παρὰ διάμετρον τὰν **ΓΔ**· φαμὶ δή, ἐπιζευχθεῖσαι αἱ **ΕΔ**, **ΔΒ** εὐθεῖαι ἐσσοῦνται ἀλλάλαις ἐπ’ εὐθείας.

Ἐπεζεύχθω γὰρ ἁ **ΖΗ** καὶ ἐκβεβλήσθω ποτὶ τὸ **Ε**, ἄχθω δὲ ἁ **ΔΘ** παρὰ τὰν **ΖΗ**.

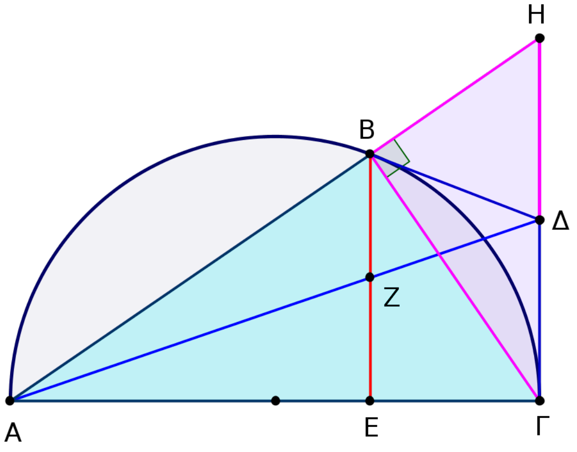
Ἐπεὶ οὖν εὐθεῖαι αἱ **ΖΒ**, **ΖΕ** ἴσαι ἐντὶ καὶ ἁ **ΗΔ** τᾷ **ΖΘ**, κοινὰ ἀφαιρήσθω ἁ **ΖΘ**, τουτέστιν ἁ **ΗΕ**· λοιπαὶ ἄρα εὐθεῖαι αἱ **ΘΔ**, **ΘΒ** ἴσαι ἀλλαλαις ἐντί.

χωνία ἄρα ἁ ὑπὸ **ΘΔΒ** χωνίᾳ τᾷ ὑπὸ **ΘΒΔ**, τουτέστιν τᾷ ὑπὸ **ΗΔΕ**, ἐστὶν ἴσα· κοινὰ ποτικείσθω χωνία ἁ ὑπὸ **ΗΔΒ** συναμφότερος ἄρα χωνία ἁ ὑπὸ **ΗΔΒ**, **ΔΒΖ** συναμφοτέρῳ τᾷ ὑπὸ **ΗΔΒ**, **ΕΔΗ** ἐστὶν ἴσα· ἔστι δὲ συναμφότερος ἁ ὑπὸ **ΗΔΒ**, **ΔΒΖ** δυσὶν ὀρθαῖς ἴσα· συναμφότερος ἄρα χωνία ἁ ὑπὸ **ΗΔΒ**, **ΕΔΗ** δυσὶν ὀρθαῖς ἐστιν ἴσα· ἐπ’ εὐθείας ἄρα ἐντὶ εὐθεῖαι αἱ **ΕΔ**, **ΔΒ**· δέδεικται οὖν τὸ προτεθέν.



β′.

Ἔστω ἁμικύκλιον τὸ ΑΒΓ καὶ δύο εὐθεῖαι ἐπιψαύουσαι αὐτοῦ αἱ ΔΒ, ΔΓ, ἁ δὲ ΒΕ ἄχθω ποτ’ ὀρθὰς τᾷ ΑΓ, ἐπεζεύχθω δὲ ἁ ΑΔ· φαμὶ δὴ τὰν ΒΖ ἴσαν εἶμεν τᾷ ΖΕ.



Ἐπεζεύχθω γὰρ ἁ **ΑΒ** καὶ ἐκβληθεῖσαι αἱ **ΑΒ**, **ΓΔ** συμπιπτέτωσαν κατὰ τὸ **Η** σαμεῖον καὶ ἄχθω ἁ **ΒΓ**.

Ἐπεὶ οὖν χωνία ἁ ὑπὸ **ΓΒΑ** ὀρθά ἐστιν, ἐσσεῖται καὶ χωνία ἁ ὑπὸ **ΓΒΗ** ὀρθά· ἔστι δὲ καὶ εὐθεῖα ἁ **ΒΔ** τᾷ **ΔΓ** ἴσα· ἐσσεῖται ἄρα καὶ εὐθεῖα ἁ **ΔΗ** τᾷ **ΔΒ**, τουτέστι τᾷ **ΔΓ** ἴσα.

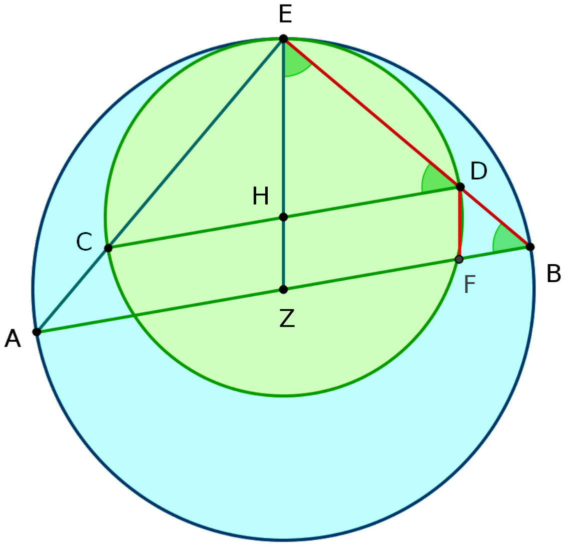
Καὶ ἐπεὶ ἁ **ΒΕ** παρὰ τὰν **ΗΓ** ἐστίν, ἐσσεῖται ἄρα καὶ ἁ **ΒΖ** τᾷ **ΖΕ** ἴσα·

δέδεικται οὖν τὸ προτεθέν.

1.

If two circles touch internally, and two diameters of these two circles are parallel, then if the contact point with the end points of the diameters are joined, the two lines will lie on a straight line.

Let **Z**, **H**, be the centres of two circles that touch internally at point **E**, and let diameter **AB** be parallel with diameter **CD**. I say that if the lines **ED**, **DB** are joined then they will lie on a straight line.



Let **ZH** be joined and produced to **E** and draw **DF** parallel to **ZH**.

Then (by substaction)

$$\left.\begin{cases} ZB = ZE \\ ZF = HD = HE \end{cases}\right\} \rightarrow FD = FB$$

It follows that

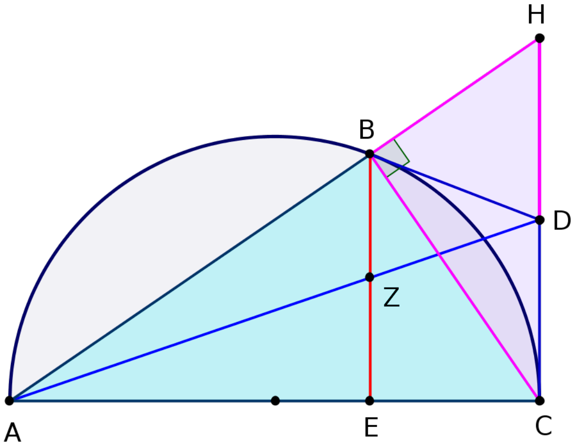
$$\left.\begin{cases} \angle FDB = \angle FBD \\ \angle HDE = \angle HED \\ \angle FBD = \angle HED \end{cases}\right\} \rightarrow$$

$$\begin{aligned} &\rightarrow \angle FDB = \angle DBZ = \angle EDH = \angle HED \\ &\rightarrow \angle HDB + \angle DBZ = \angle HDB + \angle EDH \\ &\rightarrow 2^{L} = \angle HDB + \angle EDH \end{aligned}$$

Therefore the lines **ED**, **DB** will lie on a straight line. Therefore the claim has been proved.[1]

2.

Let ABC be a semicircle and let DB, DC be the tangents from point D to the circle. Draw BE perpendicular to AC and join AD. I say that $BZ = ZE$.



Draw **AB** and produce **AB**, **CD** to meet to point **H** and draw **BC**.

Then since the $\angle ABC = 1^{L}$ follows that $\angle CBH = 1^{L}$. And $BD = DC$ so that $DH = DB = DC$.

And, since **BE** is parallel with **HC** it follows that $BZ = ZE$.
Therefore the claim has been proved.

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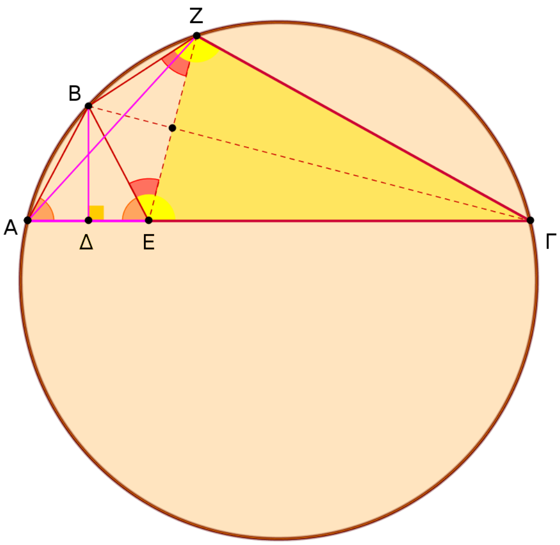
[1] The same proof applies if the circles touch externally.

γʹ.

Ἔστω τμᾶμα κύκλου τὸ ΑΓ καὶ ἀπὸ σαμείου τινος Β τᾶς περιφερείας ἄχθω τᾷ ΑΓ ποτ᾽ ὀρθὰς ἁ ΒΔ, λελάφθω δὲ εὐθεῖα ἁ ΔΕ εὐθείᾳ τᾷ ΔΑ ἴσα καὶ περιφέρεια ἁ ΒΖ τᾷ ΑΒ· φαμὶ δή, ἐπιζευχθεῖσα ἁ ΓΖ εὐθεῖα τᾷ ΓΕ ἐστὶν ἴσα.



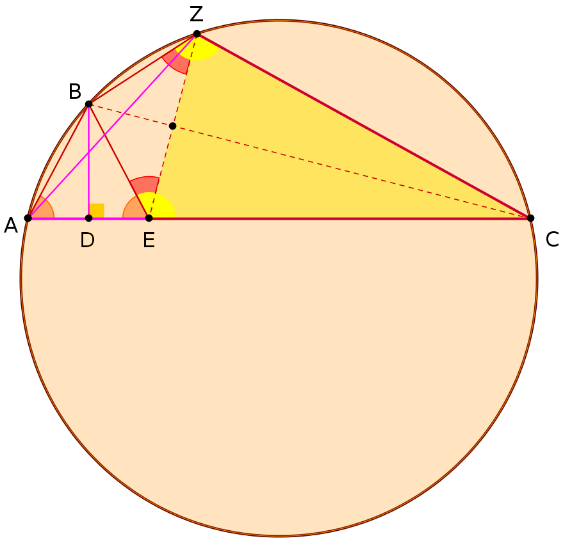
Ἐπεζεύχθωσαν γὰρ αἱ **ΑΒ**, **ΒΖ**, **ΖΕ**, **ΕΒ** εὐθεῖαι· καὶ ἐπεὶ ἁ **ΑΔ** τᾷ **ΔΕ** ἐστὶν ἴσα, κοινὰ δὲ ἁ **ΒΔ**, δύο δὴ αἱ **ΑΔ**, **ΔΒ** δυσὶ ταῖς **ΕΔ**, **ΔΒ** ἑκατέρα ἑκατέρᾳ ἴσαι ἐντί· ἔστι δὲ γωνία ἁ ὑπὸ **ΑΔΒ** γωνίᾳ τᾷ ὑπὸ **ΕΔΒ** ἴσα· βάσις ἄρα ἁ **ΕΒ** βάσει τᾷ **ΑΒ**, τουτέστι τᾷ **ΒΖ**, ἐστὶν ἴσα· γωνία ἄρα ἁ ὑπὸ **ΒΕΖ** γωνίᾳ τᾷ ὑπὸ **ΒΖΕ** ἐστὶν ἴσα.

Καὶ ἐπεὶ τετράπλευρον τὸ **ΑΒΖΓ** ἐν κύκλῳ ἐστίν, γωνίαι αἱ ἀπεναντίον αἱ ὑπὸ **ΓΖΒ**, **ΓΑΒ**, τουτέστιν αἱ ὑπὸ **ΓΖΒ**, **ΒΕΑ**, δυσὶν ὀρθαῖς ἴσαι ἐντί.

Ἔστι δὲ καὶ συναμφότερος ἁ ὑπὸ **ΓΕΒ**, **ΒΕΑ** δυσὶν ὀρθαῖς ἴσα· κοινὰ ἀφαιρήσθω ἁ ὑπὸ **ΒΕΑ**· γωνία ἄρα ἁ ὑπὸ **ΓΖΒ** γωνίᾳ τᾷ ὑπὸ **ΓΕΒ** ἐστὶν ἴσα· κοινὰ ἀφαιρήσθω ἁ ὑπὸ **ΒΖΕ**, τουτέστιν ἁ ὑπὸ **ΒΕΖ**· λοιπαὶ ἄρα αἱ ποτὶ τᾷ βάσει τᾷ **ΕΖ** τριγώνου τοῦ **ΕΓΖ** γωνίαι αἱ ὑπὸ **ΓΖΕ**, **ΖΕΓ** ἴσαι ἀλλάλαις ἐντί· πλευρὰ ἄρα ἁ **ΖΓ** πλευρᾷ τᾷ **ΕΓ** ἐστὶν ἴσα· δέδεικται οὖν τὸ προτεθέν.

3.

Let B be any point on a segment of a circle whose base is AC and let BD perpendicular to AC. Take E on AC so that $AD = DE$ and Z on arc AC so that arc AB is equal to arc BZ. I say that if CZ, CE be joined, shall be equal.



Let the lines AB, BZ, ZE, EB be joined. Then

$$\left.\begin{cases} AD = DE \\ BD \text{ common} \\ \angle ADB = \angle EDB = 1^{L} \end{cases}\right\} \rightarrow EB = AB$$

$\rightarrow EB = AB = BZ$
$\rightarrow \angle BEZ = \angle BZE$

And since $ABZC$ is an inscribed quadrilateral

$$\angle CZB + \angle CAB = 2^{L} = \angle CZB + \angle BEA$$

Hence we get

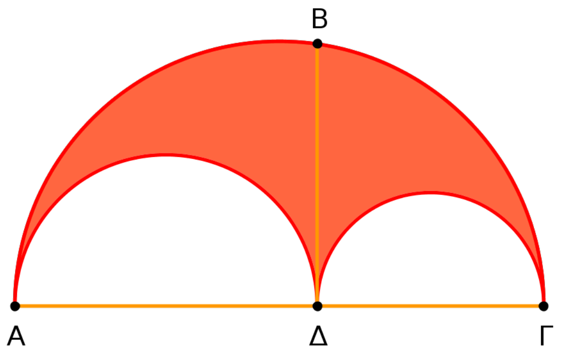
$$\left.\begin{cases} \angle CZB + \angle BEA = 2^{L} \\ \angle CEB + \angle BEA = 2^{L} \end{cases}\right\} \rightarrow$$

$\rightarrow \angle CZB = \angle CEB$
$\rightarrow \angle CZB - \angle BZE = \angle CEB - \angle BEZ$
$\rightarrow \angle CZE = \angle ZEC$

That is triangle CZE is isosceles, therefore $ZC = EC$.
Therefore the claim has been proved.

δ′.

Εἴ κα ἐν ἁμικυκλίῳ σαμεῖόν· τι ἐπὶ τᾶs διαμέτρου ᾖ, γραφέωντι δὲ ἀπὸ τῶν τμαμάτων τᾶs διαμέτρου δύο ἁμικύκλια ἐντόs, ἀναστακῇ δὲ ἀπὸ τοῦ λαφθέντοs σαμείου εὐθεῖα ποτὶ τᾷ περιφερείᾳ τᾷ διαμέτρῳ ποτ’ ὀρθάs, σχῆμα τὸ ὑπὸ τῶν τριῶν περιφερειῶν περιεχόμενον ἴσον ἐστὶ κύκλῳ, οὗ διάμετροs ἁ ἀναστακεῖσα κάθετοs.



Ἔστω ἁμικύκλιον τὸ **ΑΒΓ** καὶ σαμεῖόν τι ἐπὶ διαμέτρου τᾶs **ΑΓ** τὸ **Δ**, καὶ ἀπὸ διαμέτρων τῶν **ΓΔ**, **ΔΑ** ἁμικύκλια ἀναγεγράφθων ἐντόs, ἀπὸ δὲ τοῦ **Δ** σαμείου ἀνεστακέτω ποτ’ ὀρθὰs τᾷ **ΑΓ** ἁ **ΔΒ** φαμὶ δή, σχῆμα τὸ ὑπὸ τῶν τριῶν περιφερειῶν περιεχόμενον, τουτέστι τοῦ μείζονοs ἁμικυκλίου καὶ τῶν δύο ἀναγραφέντων ἐντόs, ὅπερ ἄρβηλοs καλείσθω, κύκλῳ, οὗ διάμετροs ἁ **ΔΒ**, ἴσον ἐστίν.

Ἐπεὶ γὰρ εὐθεῖαι αἱ **ΔΑ**, **ΔΒ**, **ΔΓ** ἑξῆs ἀνάλογόν ἐντι, ἐσσεῖται τὸ ὑπὸ τῶν **ΑΔ**, **ΔΓ** τῷ ἀπὸ τᾶs **ΒΔ** ἴσον· κοινὸν ποτικείσθω τὸ ὑπὸ τῶν **ΑΔ**, **ΔΓ** καὶ τὰ ἀπὸ τῶν **ΑΔ**, **ΔΓ**· τὸ ἄρα ἀπὸ τᾶs ὅλαs τετράγωνον, τουτέστι τὸ ἀπὸ τᾶs **ΑΓ**, τοῖs ἀπὸ τῶν τμαμάτων τῶν ἀπὸ τῶν **ΑΔ**, **ΔΓ** τετραγώνοιs καὶ τῷ δὶs τοῦ ἀπὸ τᾶs **ΒΔ** ἐστὶν ἴσον.

Καὶ ἐπεὶ οἱ κύκλοι πρὸs ἀλλάλουs ὡs τὰ ἀπὸ τᾶν διαμέτρων τετράγωνά ἐντι, ἐσσεῖται δὴ κύκλοs, οὗ διάμετροs ἁ **ΑΓ**, δυσὶ κύκλοιs, ὧν διάμετροs ἁ **ΔΒ**, καὶ δυσὶ κύκλοιs, ὧν διάμετροι αἱ **ΑΔ**, **ΔΓ**, ἴσοs, τουτέστιν ἁμικύκλιον τὸ **ΑΓ** ἴσον κύκλῳ, οὗ διάμετροs ἁ **ΔΒ**, καὶ δυσὶν ἁμικυκλίοιs, ὧν διάμετροι αἱ **ΑΔ**, **ΔΓ**· κοινὸν ἀφαιρήσθω ἁμικύκλια τὰ **ΑΔ**, **ΔΓ**· λοιπὸν ἄρα χωρίον τὸ περιεχόμενον ὑπὸ περιφερειῶν τᾶν **ΑΓ**, **ΑΔ**, **ΔΓ**, ὅπερ ἄρβηλοs καλεῖται, κύκλῳ, οὗ διάμετροs ἁ **ΔΒ**, ἐστὶν ἴσον· δέδεικται οὖν τὸ προτεθέν.

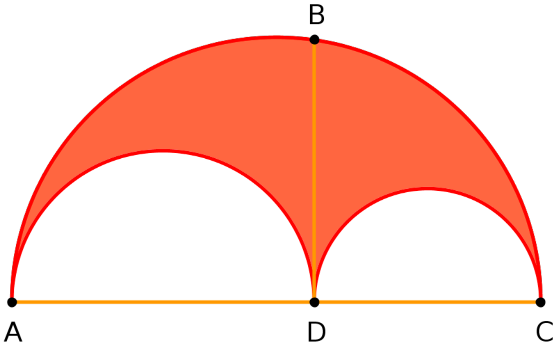
4.

If from any point on the diameter of a semicircle two inner semicircles be described with diameters the line segments defined, and from this point a perpendicular line segment is drawn and produced to meet the circumference of the outer semicircle, then the area of the figure included between the circumferences of the three semicircles is equal to the area of the circle whose diameter is the perpendicular line segment.

Let **ABC** be a semicircle and **D** any point on the diameter **AC** of the semicircle. With diameters **CD**, **DA** draw inner semicircles and from the point **D** draw a perpendicular line segment **DB** to **AC**.

I say that the area of the figure included between the circumferences of the three semicircles, which shall be called arbelos, is equal to the area of the circle whose diameter is the perpendicular line segment **DB**.

Since **DB** is the geometric mean of **DA**, **DC**, then adding the expression $\mathsf{AD}\cdot\mathsf{DC}+\mathsf{AD}^2+\mathsf{DC}^2$ we get



$$\mathsf{DB}^2=\mathsf{AD}\cdot\mathsf{DC}\rightarrow$$
$$\mathsf{DB}^2+\mathsf{AD}\cdot\mathsf{DC}+\mathsf{AD}^2+\mathsf{DC}^2=2\cdot\mathsf{AD}\cdot\mathsf{DC}+\mathsf{AD}^2+\mathsf{DC}^2\rightarrow$$
$$2\cdot\mathsf{DB}^2+\mathsf{AD}^2+\mathsf{DC}^2=(\mathsf{AD}+\mathsf{DC})^2=\mathsf{AC}^2$$

And since circles are to one another as the squares of their diameters and denoting by $(\mathsf{C}_{\mathsf{XY}})$ the area of a circle with diameter **XY** and by $(\mathsf{S}_{\mathsf{XY}})$ the area of a semicircle with diameter **XY** then we take from above

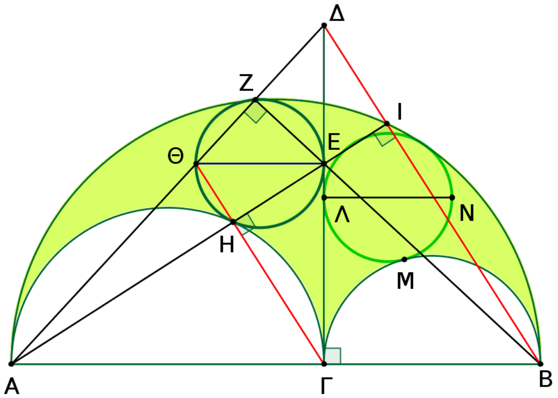
$$2\cdot(\mathsf{C}_{\mathsf{DB}})+(\mathsf{C}_{\mathsf{AD}})+(\mathsf{C}_{\mathsf{DC}})=(\mathsf{C}_{\mathsf{AC}})\rightarrow$$
$$(\mathsf{C}_{\mathsf{DB}})+(\mathsf{S}_{\mathsf{AD}})+(\mathsf{S}_{\mathsf{DC}})=(\mathsf{S}_{\mathsf{AC}})\rightarrow$$
$$(\mathsf{C}_{\mathsf{DB}})=(\mathsf{S}_{\mathsf{AC}})-(\mathsf{S}_{\mathsf{AD}})-(\mathsf{S}_{\mathsf{DC}})\rightarrow$$
$$(\mathsf{C}_{\mathsf{DB}})=(\mathsf{E}_{\mathsf{arbelos}})$$

Therefore the claim has been proved.

ε΄.

Εἴ κα ἐν ἁμικυκλίῳ σαμεῖόν τι ἐπὶ τᾶς διαμέτρου ᾖ, καὶ γραφέωντι ἀπὸ τῶν τμαμάτων τᾶς διαμέτρου δύο ἁμικύκλια ἐντός, ἀναστακῇ δὲ ἀπὸ τοῦ σαμείου εὐθεῖα τᾷ διαμέτρῳ ποτ᾽ ὀρθάς, καὶ δύο κύκλοι γραφέωντι ἐπ᾽ ἀμφότερα τᾶς ἀνεστακούσας ἐπιψαύοντες αὐτᾶς καὶ τῶν ἁμικυκλίων, οἱ γραφέντες κύκλοι ἐσσοῦνται ἀλλάλοις ἴσοι.

Ἔστω ἁμικύκλιον, οὗ διάμετρος ἁ **ΑΒ**, σαμεῖον δέ τι ἐπ᾽ αὐτᾶς τὸ **Γ**· ἀναγεγράφθω δὲ ἀπὸ τμαμάτων τῶν **ΑΓ**, **ΓΒ** ἁμικύκλια ἐντός, καὶ ἀπὸ τοῦ **Γ** σαμείου ἀνεστακέτω ποτ᾽ ὀρθὰς διαμέτρω τᾷ **ΑΒ** ἁ **ΓΔ**, γεγράφθων δὲ δύο κύκλοι ἐπ᾽ ἀμφότερα τᾶς ἀνεστακούσας εὐθείας ἐπιψαύοντες τᾶς τε ἀνεστακούσας καὶ τῶν ἁμικυκλίων· φαμὶ δή, οἱ γραφέντες κύκλοι ἴσοι ἀλλάλοις ἐντί.



Ἔστω γὰρ πρότερον κύκλος ὁ ἐπιψαύων τᾶς **ΓΔ** κατὰ τὸ **Ε** σαμεῖον καὶ ἁμικυκλίου μὲν τοῦ **ΑΓ** κατὰ τὸ **Η**, ἁμικυκλίου δὲ τοῦ **ΑΒ** κατὰ τὸ **Ζ**, ἄχθω δὲ διάμετρος τοῦ κύκλου ἁ **ΘΕ**·

ἐπιζευχθεῖσαι δὴ αἱ **ΑΘ**, **ΘΖ** εὐθεῖαι ἐσσοῦνται ἀλλάλαις ἐπ᾽ εὐθείας, ἐκβληθεῖσαι δὲ αἱ **ΑΖ**, **ΓΕ** εὐθεῖαι συμβαλέτωσαν κατὰ τὸ **Δ** σαμεῖον· ὁμοίως δὴ ἐπιζευχθεῖσαι αἱ **ΖΕ**, **ΕΒ** ἐσσοῦνται ἀλλάλαις ἐπ᾽ εὐθείας, καὶ αἱ **ΘΗ**, **ΗΓ**, καὶ αἱ **ΕΗ**, **ΗΑ**, ἐκβεβλήσθω δὲ ἁ **ΑΕ** ἐπὶ τὸ **Ι** σαμεῖον, ἄχθω δὲ ἁ **ΒΙ** εὐθεῖα καὶ ἁ **ΙΔ**.

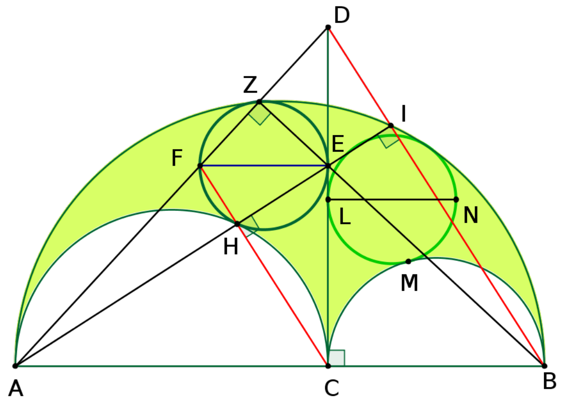
Ἐπεὶ οὖν αἱ **ΑΔ**, **ΑΒ** εὐθεῖαί ἐντι καὶ ἀπὸ τοῦ **Δ** σαμείου τᾷ **ΑΒ** ἆκται ποτ᾽ ὀρθὰς ἁ **ΔΓ**, καὶ ἀπὸ τοῦ **Β** ποτ᾽ ὀρθὰς τᾷ **ΔΑ** ἁ **ΒΖ** τεμνέουσα τὰν **ΔΓ**, κατὰ τὸ **Ε**, εὐθεῖα δὲ ἁ **ΑΕΙ** ποτ᾽ ὀρθὰς τᾷ **ΒΙ** ἐστίν, ἐσσοῦνται ἄρα εὐθεῖαι αἱ **ΒΙ**, **ΙΔ** ἀλλάλαις ἐπ᾽ εὐθείας, ὡς παῤ ἡμῶν ἐν τοῖς Περὶ ὀρθογωνίων τριγώνων δέδεικται.

Καὶ ἐπεὶ εὐθεῖα ἁ **ΒΔ** παρὰ τὰν **ΓΗ** ἐστίν, τὸν αὐτὸν λόγον ἔχει ἁ **ΑΔ** ποτὶ τὰν **ΔΘ**, ὃν ἔχει ἁ **ΑΓ** ποτὶ τὰν **ΘΕ**, τουτέστιν ἁ **ΑΒ** ποτὶ τὰν **ΒΓ** τὸ ἄρα ὑπὸ τᾶν **ΑΓ**, **ΓΒ** τῷ ὑπὸ τᾶν **ΑΒ**, **ΘΕ** ἐστὶν ἴσον· ὅμοίως δὴ δείξομες ὅτι ἐν κύκλῳ τῷ **ΛΜΝ** τὸ ὑπὸ τᾶν **ΑΓ**, **ΓΒ** τῷ ὑπὸ **ΑΒ** καὶ τᾶς διαμέτρου τοῦ **ΛΜΝ** κύκλου ἴσον ἐστίν· αἱ διάμετροι ἄρα κύκλων τῶν **ΕΖΗ**, **ΛΜΝ** ἴσαι ἐντί, τουτέστιν οἱ δύο κύκλοι ἴσοι ἐντί· δέδεικται οὖν τὸ προτεθέν.

5.

If from any point on the diameter of a semicircle two inner semicircles be described with diameters the line segments defined, and from this point a perpendicular line segment is drawn and produced to meet the circumference of the outer semicircle, then, if two circles be drawn touching the perpendicular line segment on different sides and each touching the two of the semicircles, the circles so drawn will be equal.

Let **C** be any point on the diameter **AB** of a semicircle. With diameters **AC**, **CB** draw inner semicircles and from point **C** draw **CD** perpendicular to **AB**. On each side of **CD** draw circles touching the segments and the semicircles. I say that the circles so drawn will be equal.



Consider first the circle touching **CD** at point **E**, semicircle **AC** at **H** and semicircle **AB** at **Z** and draw the diameter **FE** parallel to **AB**.

Join line segments **AF**, **FZ**. Then by Prop. 1 these will lie on straight line. Produce then line segments **AZ**, **CE** to meet at point **D**. In a similar manner by Prop. 1 line segments **ZE**, **EB** will lie on straight line as well as line segments **FH**, **HC** and line segments **EH**, **HA**. Produce also **AE** to meet circumference at **I** and draw line segments **BI** and **ID**.
Then since **AD**, **AB** are straight lines and **DC** perpendicular to **AB** and **BZ** perpendicular to **DA** meeeting **DC** at point **E**, implies **E** is the orthocenter of triangle **ABD**, so that **AEI** ⊥ **BD**. But **AEI** ⊥ **BI** therefore **BI**, **ID** will lie on straight line.

And since **BD** is parallel with **CH** we get

$$\frac{\mathsf{AD}}{\mathsf{DF}} = \frac{\mathsf{AB}}{\mathsf{BC}} = \frac{\mathsf{AC}}{\mathsf{FE}} \rightarrow$$

$$\mathsf{AC} \cdot \mathsf{CB} = \mathsf{AB} \cdot \mathsf{FE}$$

In a similar manner we can prove that in circle **LMN**

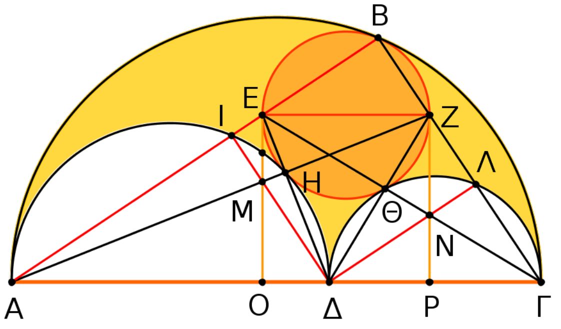
$$\mathsf{AC} \cdot \mathsf{CB} = \mathsf{AB} \cdot \mathsf{LN}$$

Therefore the diameters of the two circles **EZH**, **LMN** are equal, hence the two circles are equal. Therefore the claim has been proved.

ϛ′.

Εἴ κα ἐν ἁμικυκλίῳ σαμεῖόν τι ἐπὶ τᾶς διαμέτρου ᾖ, καὶ γραφέωντι ἀπὸ τῶν τμαμάτων τᾶς διαμέτρου δύο ἁμικύκλια ἐντός, γραφῇ δὲ ἐν τῷ ἀρβήλῳ κύκλος ἐπιψαύων τῶν τριῶν ἁμικυκλίων, τὸν λόγον τᾶς διαμέτρου τοῦ δοθέντος ἁμικυκλίου ποτὶ τὰν διάμετρον τοῦ ἐγγραφέντος κύκλου εὑρεῖν.

Ἔστω ἁμικύκλιον τὸ **ΑΒΓ**, σαμεῖον δὲ τι ἐπὶ τᾶς διαμέτρου τὸ **Δ** καὶ πεποιήσθω οὕτως, ὥστε τὸ μεῖζον τμᾶμα τὸ **ΑΔ** ἐλάσσονος τοῦ **ΔΓ** ἁμιόλιον εἶμεν, καὶ ἀπὸ τμαμάτων τῶν **ΑΔ**, **ΔΒ** ἀναγεγράφθων ἁμικύκλια, γεγράφθω δὲ ἐν τῷ ἀρβήλω κύκλος ὁ **ΕΖ** ἐπιψαύων τῶν τριῶν ἁμικυκλίων, καὶ ἄχθω διάμετρος αὐτοῦ παρὰ τὰν **ΑΓ** ἁ **ΕΖ**. Εὑρεῖν τὸν λόγον διαμέτρου τᾶς **ΑΓ** ποτὶ διάμετρον τὰν **ΕΖ**.



Ἐπεζεύχθωσαν αἱ **ΑΕ**, **ΕΒ** εὐθεῖαι καὶ αἱ **ΓΖ**, **ΖΒ**· εὐθεῖαι δή ἐντι αἱ **ΑΒ**, **ΓΒ**, ὡς ἐν τοῖς πρότερον ἐδείχθη. Ἐπεζεύχθωσαν ἔτι αἱ **ΖΗΑ**, **ΕΘΓ**· δείκνυνται δὴ αὗται εὐθεῖαι εἶμεν· ἔτι δὲ ἐπεζεύχθωσαν αἱ **ΔΕ**, **ΔΖ**, καὶ αἱ **ΔΙ**, **ΔΛ**, καὶ ἐπιζευχθεῖσαι αἱ **ΕΜ**, **ΖΝ** ἐκβεβλήσθωσαν ἐπὶ τὰ **Ο**, **Ρ** σαμεῖα.

Ἐπεὶ οὖν ἐν τριγώνῳ τῷ **ΑΕΔ** ἁ **ΑΗ** τᾷ **ΕΔ** ποτ᾽ ὀρθάς ἐστιν, καὶ ἁ **ΔΙ** τᾷ **ΑΕ**, τεμνέοντι δὲ ἀλλάλας κατὰ τὸ **Μ** σαμεῖον, ἁ **ΕΜΟ** τᾷ **ΑΓ** ἐσσεῖται ποτ᾽ ὀρθάς, ὡς παρ᾽ ἡμῶν ἐν τοῖς Περὶ τριγώνων ἐδείχθη καὶ τῷ πρότερον ὑπέκειτο· διὰ τὰ αὐτὰ δὴ καὶ ἁ **ΖΝΡ** τᾷ **ΓΑ** ἐσσεῖται ποτ᾽ ὀρθάς· ἔστι δὲ εὐθεῖα ἁ **ΔΛ** παρὰ τὰν **ΑΒ** καὶ ἁ **ΔΙ** παρὰ τὰν **ΓΒ**·

ὥστε τὸν αὐτὸν λόγον ἔχει ἁ **ΑΔ** ποτὶ τὰν **ΔΓ**, ὃν ἔχει ἁ **ΑΜ** ποτὶ τὰν **ΜΖ**, τουτέστιν ἁ **ΑΟ** ποτὶ τὰν **ΟΡ** καὶ ἁ **ΓΔ** ποτὶ τὰν **ΔΑ** τὸν αὐτὸν λόγον ἔχει, ὃν ἔχει ἁ **ΓΝ** ποτὶ τὰν **ΝΕ**, τουτέστιν ἁ **ΓΡ** ποτὶ τὰν **ΡΟ**· ἦν δὲ ἁ **ΑΔ** ἁμιόλιος τᾶς **ΔΓ**· καὶ ἁ **ΑΟ** ἄρα τᾶς **ΟΡ** ἐστὶν ἁμιόλιος, καὶ ἁ **ΟΡ** τᾶς **ΓΡ**· εὐθεῖαι ἄρα αἱ **ΑΟ**, **ΟΡ**, **ΡΓ** ἑξῆς ἀνάλόγον ἐντι, ἂν ἁ μὲν **ΡΓ** ἴσα γίνεται τέσσαρα, ἁ δὲ **ΟΡ** ἕξ, ἁ δὲ **ΑΟ** ἐννέα, ἁ δὲ **ΓΑ** ἐννεακαίδεκα.

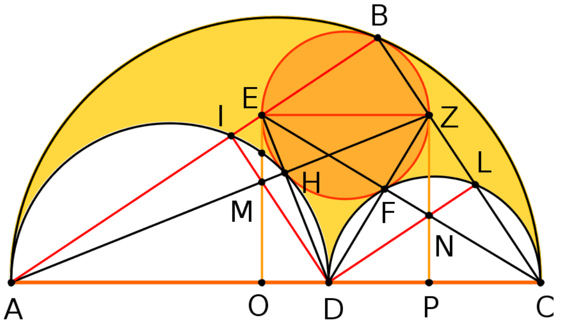
Ἔστι δὲ ἁ **ΡΟ** τᾷ **ΕΖ** ἴσα· ὥστε τὸν αὐτὸν λόγον ἔχει ἁ **ΑΓ** ποτὶ τὰν **ΕΖ**, ὃν ἔχει τὰ ἐννεακαίδεκα ποτὶ τὰ ἕξ· καί ἐστιν ἁ **ΑΓ** διάμετρος ἁμικυκλίου τοῦ **ΑΒΓ**, ἁ δὲ **ΕΖ** κύκλου τοῦ **ΕΒΖ**· εὑρέθη ἄρα ὁ αἰτούμενος λόγος.

Ὁμοίως δὴ δειχθήσεται εἴ κα ὁ λόγος τᾶς διαμέτρου τοῦ δοθέντος ἁμικυκλίου ποτὶ τὰν διάμετρον τοῦ ἐγγραφέντος κύκλου ἐπιμόριος ᾖ.

6.

If from any point on the diameter of a given semicircle, two inner semicircles be described with diameters the line segments defined, and inside the arbelos defined, a circle is drawn touching the three semicircles, find the ratio of the given semicircle's diameter to the circle's diameter.

Let **ABC** be a semicircle and **D** a point on its diameter so that $AD = (3/2)DC$. With diameters **AD**, **DC** draw inner semicircles and inside the defined arbelo draw a circle touching the three semicircles. If **EZ** is the diameter of the circle parallel with **AC**, find the ratio of **AC** to **EZ**.



Join line segments **AE**, **EB** as well as **CZ**, **ZB**. Then the line segments **AB**, **CB** as was previously shown (Prot. 1) will lie on straight line. For the same reason **ZHA**, **EFC** will also lie on a straight line. Moreover join **DE**, **DZ** and **DI**, **DL**. Also join **EM**, **ZN** and produce them until meet **AC** to points **O**, **P** respectively.

Since in triangle **AED**, **AH** is perpendicular to **ED**, and **DI** to **AE**, and they intersect at point **M**, then (as has been shown in ”About Triangles” and assumed previously) **EMO** should be perpendicular to **AC**. For the same reason **ZNP** is perpendicular to **CA**. And since **DL** is parallel to **AB** and **DI** is parallel to **CB**, we take

$$\left.\begin{array}{r} \frac{AD}{DC}=\frac{AM}{MZ}=\frac{AO}{OP}\\[4pt] \frac{CD}{DA}=\frac{CN}{NE}=\frac{CP}{PO}\\[4pt] \frac{AD}{DC}=\frac{3}{2}\end{array}\right\}\rightarrow \begin{array}{l} AO=\frac{3}{2}OP\\[8pt] OP=\frac{3}{2}CP\end{array}$$

Therefore the line segments **AO**, **OP**, **PC** are in continued proportion (geometric progression) and if $PC = 4$ then $OP = 6$, $AO = 9$ καὶ $CA = 19$. Now $PO = EZ$ so that

$$\frac{AC}{EZ}=\frac{AC}{PO}=\frac{19}{6}$$

where **AC** is the diameter of the given semicircle **ABC** and **EZ** the diameter of the circle **EBZ**. Therefore the ratio was found.

In a similar manner we find the ratio of the given semicircle's diameter to the circle's diameter when the initial ratio is of the form $\frac{n+1}{n}$.

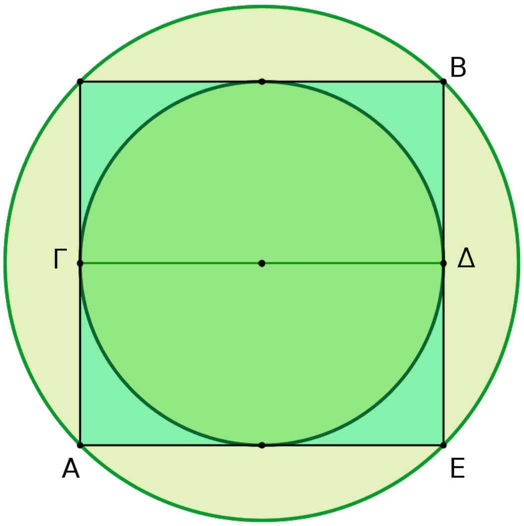
ζʹ.

Ὁ τετραγώνῳ περιγεγραμμένος κύκλος διπλασίων τοῦ ἐγγεγραμμένου ἐστίν.

Ἔστω γὰρ κύκλος ὁ **AB** περὶ τετράγωνον τὸ **AB** καὶ ἐν αὐτῷ ἐγγεγραμμένος κύκλος ὁ **ΓΔ**, διάμετρος δὲ τοῦ περιγεγραμμένου κύκλου καὶ τοῦ τετραγώνου, ἁ **AB**, ἄχθω δὲ διάμετρος τοῦ ἐγγεγραμμένου κύκλου ἁ **ΓΔ** παρὰ τὰν **AE**· φαμὶ δή, ὁ περιγεγραμμένος κύκλος τοῦ ἐγγεγραμμένου ἐστὶ διπλασίων.

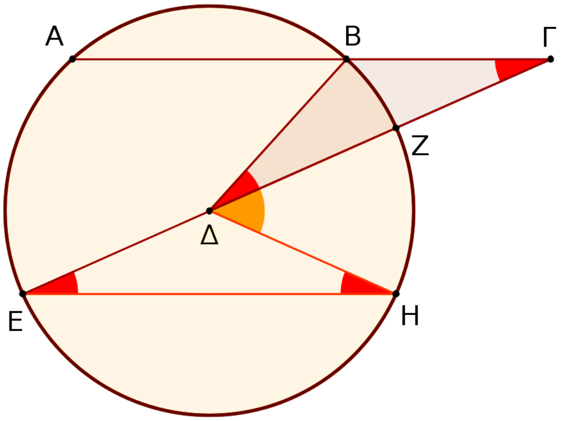
Ἐπεὶ οὖν τὸ ἀπὸ τᾶς **AB** διπλάσιον τοῦ ἀπὸ τᾶς **AE**, τουτέστι τοῦ ἀπὸ τᾶς **ΓΔ**, οἱ κύκλοι δὲ ἐντι ὡς τὰ ἀπὸ τᾶν διαμέτρων αὐτῶν τετράγωνα, ἐσσεῖται ἄρα καὶ ὁ περιγεγραμμένος κύκλος τοῦ ἐγγεγραμμένου διπλασίων·

δέδεικται οὖν τὸ προτεθέν.



ηʹ.

Εἴ κα ἐν κύκλῳ εὐθεῖά τις AB προσαρμοσμένα ᾖ, ἐκβληθῇ δὲ κατὰ τὸ Γ σαμεῖον, ὥστε τὰν BΓ εὐθεῖαν τᾷ ἐκ τοῦ κέντρου ἴσαν εἶμεν, διαχθῇ δὲ εὐθεῖά τις ἀπὸ τοῦ Γ διὰ τοῦ κέντρου τοῦ κύκλου ἐπὶ τὸ E σαμεῖον, ἐσσεῖται περιφέρεια ἁ AE περιφερείας τᾶς BZ τριπλασίων.



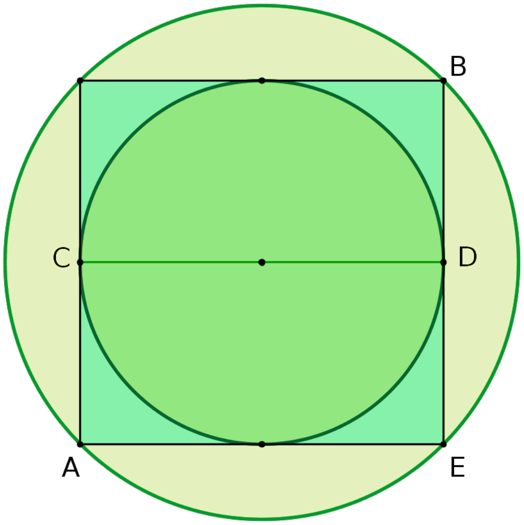
Ἄχθω γὰρ ἁ **EH** παρὰ τὰν **AB** καὶ ἐπεζεύχθων αἱ **ΔB**, **ΔH**.

Ἐπεὶ οὖν γωνίαι αἱ ὑπὸ **BΓΔ**, **BΔΓ**, **ΔEH**, **ΔHE** ἴσαι ἀλλάλαις ἐντί, γωνία δὲ ἁ ὑπὸ **ΓΔH** γωνίας τᾶς ὑπὸ **ΔEH** ἐστὶ διπλασίων, ἐσσεῖται ἄρα γωνία ἁ ὑπὸ **BΔH** γωνίας τᾶς ὑπὸ **BΔΓ** τριπλασίων.

Ἐσσεῖται ἄρα περιφέρεια ἁ **BH**, τουτέστιν ἁ **AE**, περιφερείας τᾶς **BZ** τριπλασίων. δέδεικται οὖν τὸ προτεθέν.

7.

A circle circumscribed about a square is double the circle inscribed the same square.



Let **AB** be a circle circumscribed about square **AB** and let **CD** be the circle incribed to the square so that the line segment **CD** is parallel with **AE**. I say that the circumscribed circle is double the inscribed circle.

Since we have

$$\mathbf{AB}^2 = 2 \cdot \mathbf{AE}^2 = 2 \cdot \mathbf{CD}^2$$

And since circles are to one another as the squares of their diameters, it is implied that the circumscribed circle is double the inscribed circle.
Therefore the claim has been proved.

8.

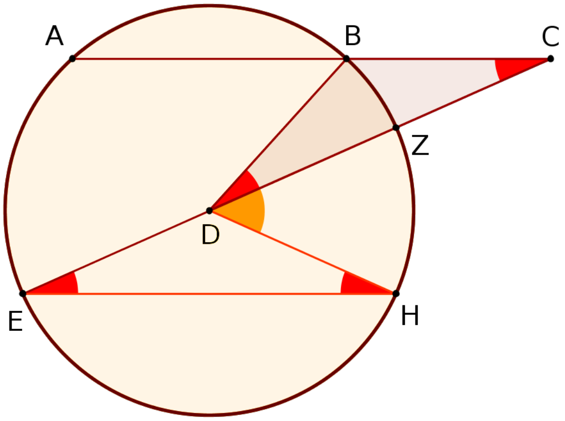
Let AB be any chord of a circle with center D be produced to point C, so that the line segment BC is equal to the radius of the circle, If further the line segment CD is produced to meet the circle a second time in E, then the arc AE will be equal to three times the arc BZ.

Draw **EH** parallel to **AB** and join **DB**, **DH**.

$$\left.\begin{array}{l}\angle\mathbf{BCD} = \angle\mathbf{BDC} = \angle\mathbf{DEH} = \angle\mathbf{DHE}\\ \angle\mathbf{CDH} = 2 \cdot \angle\mathbf{DEH}\end{array}\right\} \rightarrow$$

$$\rightarrow \angle\mathbf{BDH} = 3 \cdot \angle\mathbf{BDC}$$

Therefore the arc **BH** that is **AE** is equal to three times the arc **BZ**. Therefore the claim has been proved.



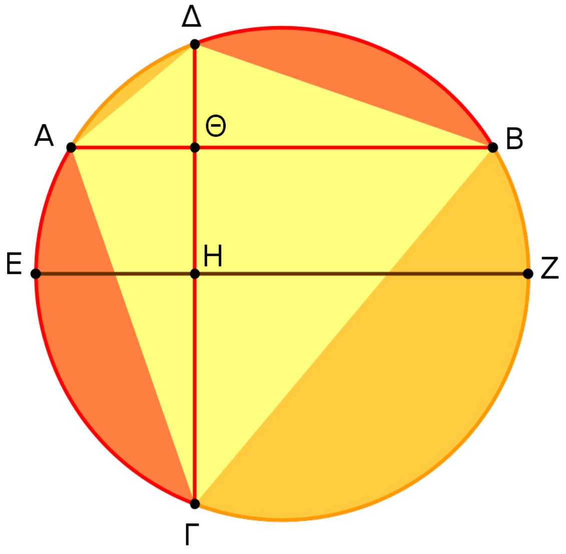
θʹ.

Εἴ κα ἐν κύκλῳ δύο εὐθεῖαι τέμνωντι ἀλλάλας ποτ' ὀρθὰς μὴ διὰ τοῦ κέντρου οὖσαι, δύο αἱ ἀπεναντίον περιφέρειαι δυσὶ ταῖς ἀπεναντίον ἴσαι ἀλλάλαις ἐντί.

Ἔστω κύκλος ὁ **ΑΒΓ** καὶ δύο εὐθεῖαι αἱ **ΑΒ**, **ΓΔ** τεμνέουσαι ἀλλάλας ποτ' ὀρθάς, μὴ διὰ τοῦ κέντρου οὖσαι· φαμὶ δή, δύο αἱ ἀπεναντίον περιφέρειαι αἱ **ΑΔ**, **ΓΒ** δυσὶ ταῖς ἀπεναντίον ταῖς **ΑΓ**, **ΒΔ** ἴσαι ἀλλάλαις ἐντί.

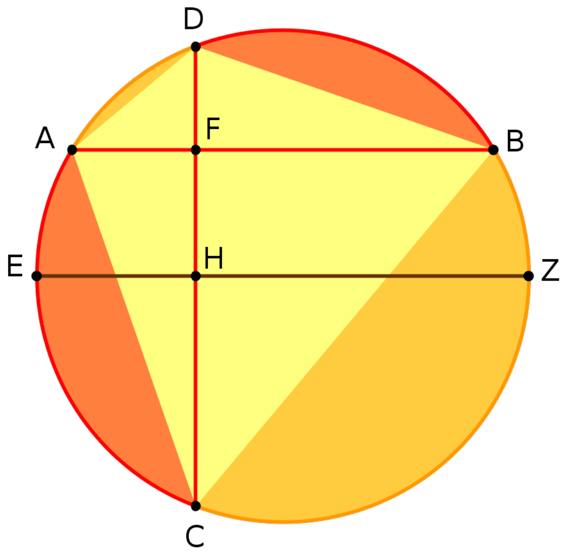
Τετμάσθω γὰρ δίχα ἁ **ΓΔ** κατὰ τὸ **Η** σαμεῖον καὶ διὰ τοῦ **Η** διάχθω διάμετρος τοῦ κύκλου ἁ **ΕΖ** παρὰ τὰν **ΑΒ**.

Ἐπεὶ οὖν περιφέρεια ἁ **ΕΓ** περιφερείαις ταῖς **ΕΑ**, **ΑΔ** ἴσα ἐστίν, ἐσσοῦνται ἄρα περιφέρειαι αἱ **ΓΖ**, **ΕΑ**, **ΑΔ** ἁμικυκλίῳ ἴσαι. ἔστι δὲ περιφέρεια ἁ **ΕΑ** περιφερείᾳ τᾷ **ΒΖ** ἴσα· συναμφότερος ἄρα περιφέρεια ἁ **ΓΒ**, **ΑΔ** ἁμικυκλίῳ ἐστὶν ἴσα· λοιπὴ ἄρα συναμφότερος περιφέρεια ἁ **ΑΓ**, **ΔΒ** ἁμικυκλίῳ ἐστὶν ἴσα· δέδεικται οὖν τὸ προτεθέν.



9.

If in a circle two chords intersect at right angles and do not pass through the centre then the sum of two opposite arcs equals the sum of the remaining others.



Let **ABC** be a circle and let **AB**, **CD** be two chords which intersect at right angles and which do not pass through the centre. I say that the sum of the two opposite arcs $\overset{\frown}{AD}$, $\overset{\frown}{CB}$ is equal to the sum of the remaining others $\overset{\frown}{AC}$, $\overset{\frown}{BD}$. That is

$$\overset{\frown}{AD} + \overset{\frown}{CB} = \overset{\frown}{AC} + \overset{\frown}{BD}$$

Let **H** be the midpoint of **CD** and from point **H** draw the diameter of the circle **EZ** parallel with chord **AB**.

Since

$$\overset{\frown}{EC} = \overset{\frown}{EA} + \overset{\frown}{AD}$$
$$\rightarrow \overset{\frown}{CZ} + \overset{\frown}{EA} + \overset{\frown}{AD} = 1 \text{ semicircle}$$

However $EA = BZ$ therefore

$$\rightarrow \overset{\frown}{CZ} + \overset{\frown}{BZ} + \overset{\frown}{AD} = 1 \text{ semicircle}$$
$$\rightarrow \overset{\frown}{CB} + \overset{\frown}{AD} = 1 \text{ semicircle}$$

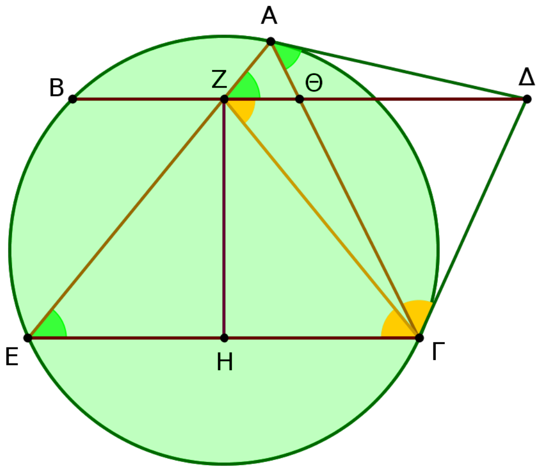
This implies that the sum of the remaining arcs will also be equal to a semicircle, that is :

$$\overset{\frown}{AC} + \overset{\frown}{DB} = 1 \text{ semicircle}$$

Therefore the claim has been proved.

ι΄.

Εἴ κα ᾖ κύκλος ὁ ΑΒΓ καὶ δύο εὐθεῖαι αἱ ΔΑ, ΔΓ ἐπιψαύουσαι αὐτοῦ κατὰ τὰ Α, Γ σαμεῖα, τεμνέουσα δὲ εὐθεῖα ἁ ΔΒ, ἀχθῇ δὲ ἁ ΕΓ παρὰ τὰν ΒΔ, ἐπιζευχθῇ δὲ ἁ ΕΑ τεμνέουσα τὰν ΔΒ κατὰ τὸ Ζ, καὶ ἀπὸ τοῦ Ζ ποτ᾽ ὀρθὰς τᾷ ΕΓ ἀχθῇ ἁ ΖΗ, ἁ ἀχμένα τὰν ΕΓ δίχα τέμνει.

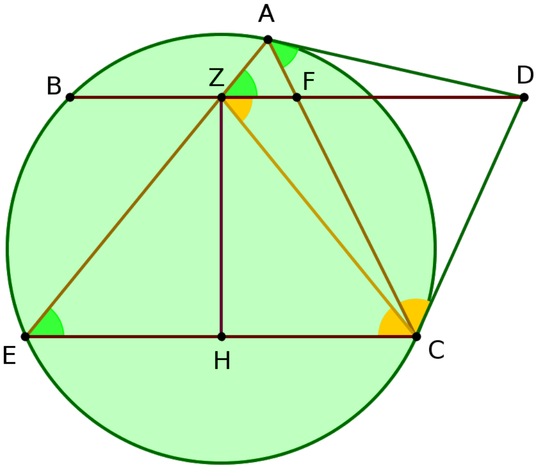


Ἐπεζεύχθω γὰρ ἁ **ΑΓ**. Ἐπεὶ οὖν εὐθεῖα ἁ **ΔΑ** ἐπιψαύουσα τοῦ κύκλου ἐστίν, ἁ δὲ **ΑΓ** τεμνέουσα αὐτοῦ, χωνία ἁ ὑπὸ **ΔΑΓ** τᾷ ἐν τῷ ἐναλλὰξ τμάματι τοῦ κύκλου χωνίᾳ τᾷ ὑπὸ **ΑΕΓ**, τουτέστι τᾷ ὑπὸ **ΑΖΔ**, ἐστὶν ἴσα. Ἔστι γὰρ ἁ **ΓΕ** παρὰ τὰν **ΒΔ**.

Καὶ ἐπεὶ ἐν δυσὶ τριχώνοις τοῖς **ΔΑΖ**, **ΑΘΔ** δύο χωνίαι αἱ ὑπὸ **ΑΖΔ**, **ΘΑΔ** ἴσαι ἀλλάλαις ἐντί, χωνία δὲ ἁ πρὸς τῷ **Δ** κοινά, τὸ ἄρα ὑπὸ τῶν **ΖΔ**, **ΔΘ** περιεχόμενον ὀρθοχώνιον τῷ ἀπὸ τᾶς **ΔΑ**, τουτέστι τῷ ἀπὸ τᾶς **ΔΓ** τετραχώνῳ, ἐστὶν ἴσον· ἐπεὶ οὖν ὃν λόχον ἔχει ἁ **ΖΔ** ποτὶ τὰν **ΔΓ**, τοῦτον ἔχει καὶ ἁ **ΔΓ** ποτὶ τὰν **ΔΘ**, χωνία δὲ ἁ ποτὶ τὸ **Δ** σαμεῖον κοινά ἐστιν, τρίχωνα ἄρα τὰ **ΔΖΓ**, **ΔΓΘ** ἐστὶν ὅμοια καὶ χωνίαι αἱ ὑπὸ **ΔΖΓ**, **ΔΓΘ**, **ΔΑΘ**, **ΑΖΔ** ἴσαι ἀλλάλαις ἐντί· ἔστι δὲ καὶ χωνία ἁ ὑπὸ **ΔΖΓ** τᾷ ὑπὸ **ΖΓΕ** ἴσα· ἦν δὲ καὶ ἁ ὑπὸ **ΔΖΑ** τᾷ ὑπὸ **ΑΕΓ** ἴσα· ἐν δυσὶ τριχώνοις ἄρα τοῖς **ΕΗΖ**, **ΓΗΖ** δύο χωνίαι αἱ ὑπὸ **ΗΕΖ**, **ΗΓΖ** ἴσαι ἀλλάλαις ἐντὶ καὶ αἱ ποτὶ τῷ **Η** σαμείῳ χωνίαι ὀρθαί· ἔστι δὲ πλευρὰ ἁ **ΗΖ** κοινά· ἔστιν ἄρα ἁ **ΕΗ** τᾷ **ΗΓ** ἴσα· δέδεικται οὖν τὸ προτεθέν.

10.

Let ABC be a circle and let DA, DC be two tangents to the circle while BD a line segment cutting it. Draw chord EC parallel with BD and join EA which meets DB at Z. Then, if ZH be drawn perpendicular to EC, it will bisect it in H.



Join AC. Since DA is tangent to the circle and AC a chord of it, and since CE is parallel with BD, it follows that

$$\angle\mathsf{DAC}=\angle\mathsf{AEC}=\angle\mathsf{AZD}$$

Then the triangles DAZ, AFD have $\angle\mathsf{AZD}=\angle\mathsf{FAD}$ and $\angle\mathsf{\Delta}$ common, they are therefore similar, hence we get

$$\begin{aligned}&\triangle\mathsf{DAZ}\approx\triangle\mathsf{AFD}\rightarrow\\&\frac{\mathsf{ZD}}{\mathsf{DA}}=\frac{\mathsf{DA}}{\mathsf{DF}}\rightarrow\\&\mathsf{ZD}\cdot\mathsf{DF}=\mathsf{DA}^2\rightarrow\\&\mathsf{ZD}\cdot\mathsf{DF}=\mathsf{DC}^2\rightarrow\\&\frac{\mathsf{ZD}}{\mathsf{DC}}=\frac{\mathsf{DC}}{\mathsf{DF}},\ \angle\mathsf{D}\text{ common}\rightarrow\\&\triangle\mathsf{DZC}\approx\triangle\mathsf{DCF}\rightarrow\\&\angle\mathsf{DZC}=\angle\mathsf{DCF}=\angle\mathsf{DAF}=\angle\mathsf{AZD}\end{aligned}$$

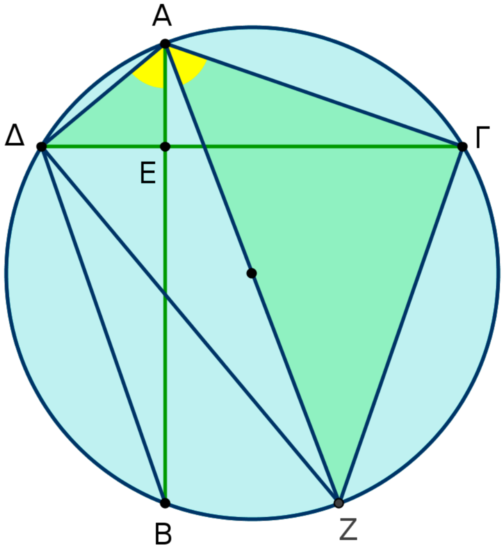
From the above and since $\mathsf{EC}\parallel\mathsf{BD}$, we get

$$\left.\begin{aligned}\angle\mathsf{DZC}=\angle\mathsf{ZCE}\\\angle\mathsf{DZA}=\angle\mathsf{AEC}\\\angle\mathsf{DZC}=\angle\mathsf{DZA}\end{aligned}\right\}\rightarrow\angle\mathsf{AEC}=\angle\mathsf{ZCE}\rightarrow\angle\mathsf{HEZ}=\angle\mathsf{HCZ}$$

Therefore the triangles EHZ, CHZ have $\angle\mathsf{HEZ}=\angle\mathsf{HCZ}$, $\angle\mathsf{ZHE}=\angle\mathsf{ZHC}=1^{\mathsf{L}}$ καὶ ZH common. It follows that they are equal and hence $\mathsf{EH}=\mathsf{HC}$. Therefore the claim has been proved.

ια΄.

Εἴ κα ἐν κύκλῳ δύο εὐθεῖαι τέμνωντι ἀλλάλας ποτ᾽ ὀρθὰς μὴ διὰ τοῦ κέντρου οὖσαι, τὰ ἀπὸ τῶν τμαμάτων τῶν εὐθειῶν τετράγωνα τῷ ἀπὸ τᾶς διαμέτρου τοῦ κύκλου ἴσα ἐντί.



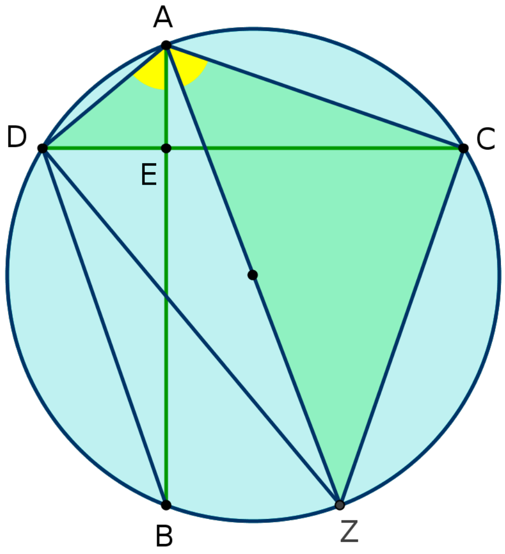
Ἔστω γὰρ κύκλος ὁ **ΑΒΓ**, καὶ δύο εὐθεῖαι αἱ **ΑΒ**, **ΓΔ** τετμάσθων ποτ᾽ ὀρθὰς κατὰ τὸ **Ε** σαμεῖον· φαμὶ δή, τὰ ἀπὸ τῶν τμαμάτων τῶν **ΑΕ**, **ΕΒ**, **ΓΕ**, **ΕΔ** τετράγωνα τῷ ἀπὸ τᾶς διαμέτρου τοῦ κύκλου ἴσα ἐστίν.

Ἄχθω γὰρ διάμετρος τοῦ κύκλου ἁ **ΑΖ** καὶ ἐπεζεύχθων αἱ **ΑΓ**, **ΑΔ**, **ΓΖ**, **ΔΒ** εὐθεῖαι.
Ἐπεὶ οὖν ἐν δυσὶ τριγώνοις τοῖς **ΑΔΕ**, **ΑΖΓ** γωνίαι αἱ ὑπὸ **ΑΕΔ**, **ΑΔΕ**, καὶ **ΑΓΖ**, **ΑΖΓ** ἴσαι ἀλλάλαις ἐντὶ ἑκατέρα ἑκατέρᾳ, λοιπαὶ ἄρα γωνίαι αἱ ὑπὸ **ΓΑΖ**, **ΔΑΕ** ἐσσοῦνται ἀλλάλαις ἴσαι· περιφέρειαι ἄρα αἱ **ΓΖ**, **ΔΒ** ἴσαι ἀλλάλαις ἐντί, καὶ αἱ ταύτας ὑποτείνουσαι εὐθεῖαι αἱ **ΓΖ**, **ΔΒ**· ἔστι δὲ καὶ τὰ ἀπὸ τῶν **ΔΕ**, **ΕΒ** τῷ ἀπὸ τᾶς **ΔΒ**, τουτέστι τῷ ἀπὸ τᾶς **ΓΖ**, ἴσον, καὶ τὰ ἀπὸ τῶν **ΑΕ**, **ΕΓ** τῷ ἀπὸ τᾶς **ΓΑ**, καὶ τὰ ἀπὸ τῶν **ΓΖ**, **ΓΑ** τῷ ἀπὸ τᾶς **ΖΑ**, τουτέστι τῷ ἀπὸ τᾶς διαμέτρου, ἴσα·

ἐσσοῦνται ἄρα τὰ ἀπὸ τῶν τμαμάτων τῶν **ΑΕ**, **ΕΒ**, **ΓΕ**, **ΕΔ** τετράγωνα τῷ ἀπὸ τᾶς διαμέτρου ἴσα· δέδεικται οὖν τὸ προτεθέν.

11.

If in a circle two chords intersect at right angles in a point not being the centre, then the sum of the squares of the defined segments is equal to the square of the diameter.



Let **ABC** be a circle and let **AB**, **CD** be two chords which intersect at right angles in point **E**. I say that

$$\mathbf{AE}^2 + \mathbf{EB}^2 + \mathbf{CE}^2 + \mathbf{ED}^2$$

is equal to the square of the diameter.

Draw the diameter **AZ** and join line segments **AC**, **AD**, **CZ**, **DB**.
Then from triangle **ADE**, **AZC** we get

$$\left.\begin{array}{l}\angle \mathbf{AED} = \angle \mathbf{ACZ}\\ \angle \mathbf{ADE} = \angle \mathbf{AZC}\end{array}\right\} \rightarrow \angle \mathbf{DAE} = \angle \mathbf{CAZ}$$

$$\rightarrow \overset{\frown}{\mathbf{DB}} = \overset{\frown}{\mathbf{CZ}}$$
$$\rightarrow \mathbf{DB} = \mathbf{CZ}$$

Therefore

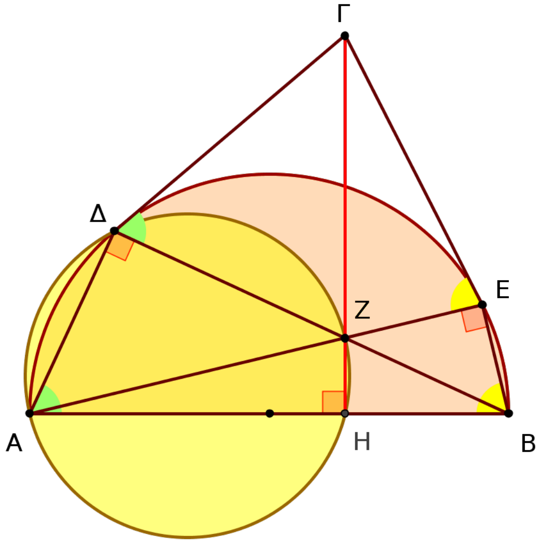
$$\left.\begin{array}{l}\mathbf{DE}^2 + \mathbf{EB}^2 = \mathbf{DB}^2 = \mathbf{CZ}^2\\ \mathbf{AE}^2 + \mathbf{EC}^2 = \mathbf{CA}^2\\ \mathbf{CZ}^2 + \mathbf{CA}^2 = \mathbf{ZA}^2\end{array}\right\} \rightarrow$$

$$\rightarrow \mathbf{AE}^2 + \mathbf{EB}^2 + \mathbf{CE}^2 + \mathbf{ED}^2 = \mathbf{ZA}^2$$

That is the sum of the squares of the defined segments is equal to the square of the diameter. Therefore the claim has been proved.

ιβ′.

Εἴ κα ἐκ σαμείου ἐκτὸς ἁμικυκλίου δύο εὐθεῖαι ἀχθέωντι ἐπιψαύουσαι αὐτοῦ, ἀχθέωντι δὲ ἐκ τῶν σαμείων ἁφᾶς δύο εὐθεῖαι ποτὶ τοῖς ἀπεναντίον περάττεσι τᾶς διαμέτρου τεμνέουσαι ἀλλάλας, ἁ ἐκ τοῦ ἐκτὸς σαμείου ποτὶ τὸ σαμεῖον τομᾶς τῶν δύο εὐθειῶν ἀχθεῖσα καὶ ἐκβληθεῖσα ποτὶ τὰν διάμετρον ἐσσεῖται ταύτᾳ ποτ’ ὀρθάς.



Ἔστω ἁμικύκλιον τὸ **ΑΒ**, σαμεῖον δὲ τι ἐκτὸς αὐτοῦ τὸ **Γ**, καὶ ἐκ τοῦ **Γ** ἄχθων δύο εὐθεῖαι αἱ **ΓΔ**, **ΓΕ** ἐπίψαύουσαι αὐτοῦ κατὰ τὰ **Δ**, **Ε** σαμεῖα, ἐπεζεύχθων δὲ ἐκ τῶν σαμείων ἁφᾶς ποτὶ τοῖς ἀπεναντίον περάττεσι τᾶς διαμέτρου τὰ **Α**, **Β** εὐθεῖαι αἱ **ΕΑ**, **ΔΒ** τεμνέουσαι ἀλλάλας κατὰ τὸ **Ζ**, καὶ ἀχθεῖσα ἁ **ΓΖ** ἐκβεβλήσθω ἐπὶ τὸ **Η** σαμεῖον· φαμὶ δή, εὐθεῖα ἁ **ΓΗ** διαμέτρῳ τᾷ **ΑΒ** ἐσσεῖται ποτ’ ὀρθάς.

Ἐπεζεύχθων γὰρ αἱ **ΑΔ**, **ΕΒ**. Ἐπεὶ οὖν τριγώνου τοῦ **ΔΑΒ** γωνία ἁ ὑπὸ **ΑΔΒ** ὀρθά ἐστιν, λοιπαὶ ἄρα γωνίαι αἱ ὑπὸ **ΔΑΒ**, **ΔΒΑ** μιᾷ ὀρθᾷ ἴσαι ἐντί· ἔστι δὲ καὶ γωνία ἁ ὑπὸ **ΑΕΒ** μιᾷ ὀρθᾷ ἴσα· κοινὰ ποτικείσθω ἁ ὑπὸ **ΖΒΕ**· συναμφότερος ἄρα ἁ ὑπὸ **ΔΑΒ**, **ΑΒΕ** συναμφοτέρῳ τᾷ ὑπὸ **ΖΒΕ**, **ΖΕΒ**, τουτέστιν ἐξωτερικᾷ γωνίᾳ τᾷ ὑπὸ **ΔΖΕ** τριγώνου τοῦ **ΖΒΕ** ἐστὶν ἴσα.

Καὶ ἐπεὶ εὐθεῖα ἁ **ΓΔ** ἐπιψαύουσα τοῦ κύκλου ἐστίν, διᾶκται δὲ ἀπὸ τοῦ σαμείου ἁφᾶς τοῦ **Δ** ἁ **ΔΒ** τεμνέουσα τὸν κύκλον, ἐσσεῖται γωνία ἁ ὑπὸ **ΓΔΒ** γωνίᾳ τᾷ ὑπὸ **ΔΑΒ** ἴσα· διὰ τὰ αὐτὰ δὴ καὶ γωνία ἁ ὑπὸ **ΓΕΖ** τᾷ ὑπὸ **ΕΒΑ** ἐστὶν ἴσα καὶ συναμφότερος ἄρα γωνία ἁ ὑπὸ **ΓΕΖ**, **ΓΔΖ** τᾷ ὑπὸ **ΔΖΕ** ἐστὶν ἴσα·

καὶ δέδεικται παῤ ἡμῶν ἐν τοῖς Περὶ τετραπλεύρων ὅτι εἴ κα μεταξὺ δύο ἴσων εὐθειῶν τεμνομένων, οἷον τῶν **ΓΔ**, **ΓΕ**, δύο εὐθεῖαι ἀχθέωντι τεμνόμεναι, οἷον αἱ **ΔΖ**, **ΕΖ**, γωνία δὲ ἁ ὑπὸ τούτων περιεχομὲνα, ὡς ἁ ποτὶ τῷ **Ζ**, συναμφοτέρῳ τᾷ ὑπὸ τῶν δύο τεμνομένων εὐθειῶν περιεχομένᾳ, ὡς αἱ ποτὶ τοῖς **Ε**, **Δ** σαμείοις, ἴσα ἐστίν, ἁ ἐπιζευχνυμένα ἐκ τοῦ σαμείου καθ’ ὃ αἱ δύο εὐθεῖαι συμβαλέοντι ἐπὶ τὸ σαμεῖον καθ’ ὃ αὗται τεμνέοντι ἀλλάλας, ὡς ἁ **ΓΖ** εὐθεῖα, ἑκατέρᾳ τῶν τεμνομένων εὐθειῶν, ὡς αἱ **ΓΔ**, **ΓΕ**, ἐστὶν ἴσα·

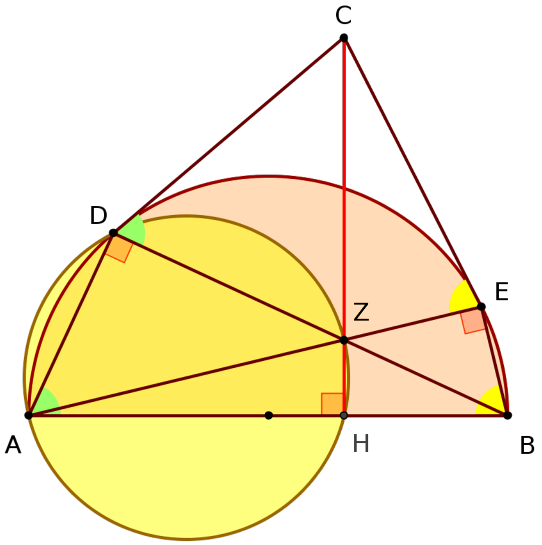
ἁ **ΓΖ** εὐθεῖα ἄρα τᾷ **ΓΔ** ἐστὶν ἴσα καὶ γωνία ἁ ὑπὸ **ΓΖΔ** γωνίᾳ τᾷ ὑπὸ **ΓΔΖ**, τουτέστι τᾷ ὑπὸ **ΔΑΗ**· γωνίαι δὲ αἱ ὑπὸ **ΓΖΔ**, **ΔΖΗ** δυσὶν ὀρθαῖς ἴσαι ἐντί· συναμφότερος ἄρα γωνία ἁ ὑπὸ **ΔΑΗ**, **ΔΖΗ** δυσὶν ὀρθαῖς ἴσα ἐστίν· λοιπαὶ ἄρα γωνίαι τετραπλεύρου τοῦ **ΑΔΖΗ** αἱ ὑπὸ **ΑΔΖ**, **ΑΗΖ** δυσὶν ὀρθαῖς ἴσαι ἐντί· ἔστι δὲ γωνία ἁ ὑπὸ **ΑΔΒ** μιᾷ ὀρθᾷ ἴσα· γωνία ἄρα ἁ ὑπὸ **ΑΗΓ** μιᾷ ὀρθᾷ ἴσα ἐστίν· ἔστιν ἄρα εὐθεῖα ἁ **ΓΗ** διαμέτρῳ τᾷ **ΑΒ** ποτ’ ὀρθάς· δέδεικται οὖν τὸ προτεθέν.

12.

If from a point outside of a semicircle, draw two tangents and join the tangency points with the opposite ends of the semicircle's diameter intersecting in a point, then the line segment which joins the point outside with the intersection point and produced to meet the diameter is perpendicular to the diameter.

Let **AB** be the diameter of a semicircle and let **CD**, **CE** be the tangents to it from any point **C**. Join the tangency points **D**, **E** with the opposite ends of the diameter **AB** with the segments **EA**, **DB** and let **Z** be the intersection point. Then, if draw **CZ** and produce it to meet **AB** at **H**, I say that the line segment **CH** is perpendicular to **AB**.

Join **AD**, **EB**. Then we take



$\angle \mathsf{DAB} + \angle \mathsf{DBA} = \angle \mathsf{AEB} \rightarrow$

$\angle \mathsf{DAB} + \angle \mathsf{DBA} + \angle \mathsf{ZBE} = \angle \mathsf{AEB} + \angle \mathsf{ZBE} \rightarrow$

$\angle \mathsf{DAB} + \angle \mathsf{EBA} = \angle \mathsf{DZE}$

but

$\angle \mathsf{CDB} = \angle \mathsf{DAB}$ καὶ $\angle \mathsf{CEZ} = \angle \mathsf{EBA}$

therefore

$\angle \mathsf{CDZ} + \angle \mathsf{CEZ} = \angle \mathsf{DAB} + \angle \mathsf{EBA} = \angle \mathsf{DZE}$

and as it is proved in the treatise "About Quadrilaterals", that if from the ends of two equal line segments like **CD**, **CE** two other line segments are drawn like **DZ**, **EZ** which meet in **Z**, then it will hold

$\angle \mathsf{CDZ} + \angle \mathsf{CEZ} = \angle \mathsf{DZE} \rightarrow \mathsf{CD} = \mathsf{CE} = \mathsf{CZ}$

Then it will hold also

$\angle \mathsf{CDZ} = \angle \mathsf{CZD} = \angle \mathsf{DAH} \rightarrow$

$\angle \mathsf{DAH} + \angle \mathsf{DZH} = \angle \mathsf{CZD} + \angle \mathsf{DZH} = 2^{\mathsf{L}} \rightarrow$

DAHZ is inscribed →

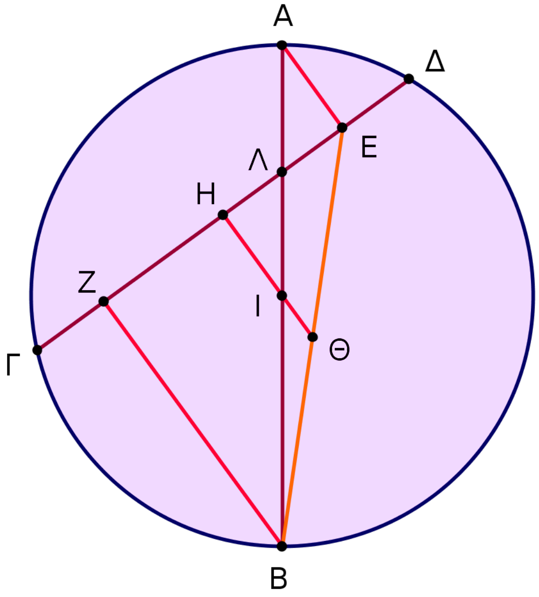
$\angle \mathsf{ADZ} + \angle \mathsf{AHZ} = 2^{\mathsf{L}} \rightarrow$

$\angle \mathsf{AHZ} = 1^{\mathsf{L}}$

That is **CH** is perpendicular to diameter **AB**. Therefore the claim has been proved.

ιγ΄.

Εἴ κα ἐν κύκλῳ δύο εὐθεῖαι τεμνέουσαι ἀλλάλας μὴ ποτ᾽ ὀρθὰς ὦσιν, ἁ μὲν διάμετρος ἁ δὲ οὔ, ἀχθέωντι δὲ ἀπὸ τῶν περάτων τᾶς διαμέτρου εὐθεῖαι ποτ᾽ ὀρθὰς τᾷ ἄλλᾳ εὐθεία, αἱ ἀπολαφθεῖσαι ἀπὸ τῶν περάτων τᾶς διαμέτρου εὐθεῖαι ἴσαι ἀλλάλαις ἐντί.



Ἔστω κύκλος ὁ **ΑΒΓ** καὶ ἐν αὐτῷ δύο εὐθεῖαι τεμνέουσαι ἀλλάλας μὴ ποτ᾽ ὀρθὰς αἱ **ΑΒ**, **ΓΔ**, ἆν ἁ **ΑΒ** διάμετρος τοῦ κύκλου, καὶ ἀπὸ τῶν περάτων τᾶς διαμέτρου τῶν **Α**, **Β** ἄχθωσαν τᾷ **ΓΔ** ποτ᾽ ὀρθὰς εὐθεῖαι αἱ **ΑΕ**, **ΒΖ**. φαμὶ δή, αἱ ἀπὸ τῶν περάτων τᾶς διαμέτρου ἀπολαφθεῖσαι εὐθεῖαι αἱ **ΓΖ**, **ΔΕ** ἴσαι ἀλλάλαις ἐντί.

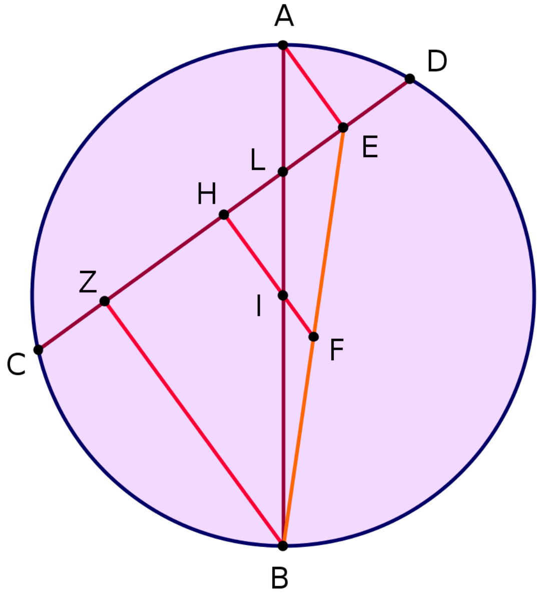
Ἐπεζεύχθω γὰρ ἁ **ΕΒ** καὶ ἀπὸ κέντρου τοῦ κύκλου τοῦ **Ι** τᾷ **ΓΔ** ἄχθω ποτ᾽ ὀρθὰς εὐθεῖα ἁ **ΙΗ** καὶ ἐκβληθεῖσα συμβαλλέτω τᾷ **ΕΒ** κατὰ τὸ **Θ** σαμεῖον.

Ἐπεὶ οὖν εὐθεῖα ἁ **ΙΗ** παρὰ τὰν **ΑΕ** ἐστίν, ἁ δὲ **ΒΙ** τᾷ **ΙΑ** ἴσα, εὐθεῖα ἄρα ἁ **ΒΘ** τᾷ **ΘΕ** ἐστὶν ἴσα.

Πάλιν, ἐπεὶ ἁ **ΒΖ** παρὰ τὰν **ΘΗ** ᾖ, ἐστίν εὐθεῖα ἄρα ἁ **ΖΗ** εὐθείᾳ τᾷ **ΗΕ** ἐστὶν ἴσα. ἔστι δὲ καὶ ἁ **ΗΓ** τᾷ **ΗΔ** ἴσα. κοινὰ ἀφαιρήσθω ἁ **ΖΗ**, τουτέστιν ἁ **ΗΕ**· λοιπὰ ἄρα ἁ **ΖΓ** λοιπᾷ τᾷ **ΕΔ** ἐστὶν ἴσα· φανερὸν οὖν ὃ ἔδει δεῖξαι.

13.

If in a circle, a diameter meets a chord not at right angles, and from the end points of the diameter draw perpendicular line segments to the chord, then the outer segments defined on the chord are equal.



Let **ABC** be a circle and let **AB**, **CD** be two chords of the circle which intersect not at right angles of whom **AB** is the diameter. Then from the end points **A**, **B** of the diameter draw perpendicular line segments **AE**, **BZ** to **CD**. I say that the line segment **CZ** is equal to line segment **DE**.

Join **EB** and from the centre **I** of the circle draw **IH** perpendicular to **CD** and produce it to meet **EB** at point **F**. Then we get

$$\left.\begin{array}{l}\mathbf{IH} \parallel \mathbf{AE}\\ \mathbf{BI} = \mathbf{IA}\end{array}\right\} \rightarrow \mathbf{BF} = \mathbf{FE}$$

Again, since

$$\left.\begin{array}{l}\mathbf{BZ} \parallel \mathbf{FH}\\ \mathbf{BF} = \mathbf{FE}\end{array}\right\} \rightarrow \mathbf{ZH} = \mathbf{HE}$$

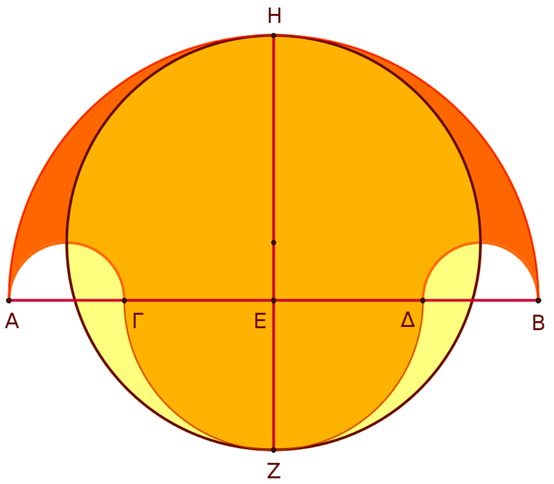
so finally

$$\left.\begin{array}{l}\mathbf{HC} = \mathbf{HD}\\ \mathbf{ZH} = \mathbf{HE}\end{array}\right\} \rightarrow \mathbf{CZ} = \mathbf{EZ}$$

It is obvious then that the claim has been proved.

ιδʹ.

Εἴ κα ἐν ἁμικυκλίῳ ἀπὸ τῶν περάτων τᾶς διαμέτρου δύο ἴσα τμάματα λαφθέωντι καὶ ἀπὸ τούτων ἁμικύκλια ἐντὸς γραφέωντι, γραφῇ δὲ ἀπὸ τοῦ λοιποῦ τμάματος τᾶς διαμέτρου ἁμικύκλιον ἐκτός, ὁ κύκλος, οὗ διάμετρος συναμφότερος ἁ ἐκ τοῦ κέντρου τοῦ ἁμικυκλίου καὶ ἁ ἐκ τοῦ κέντρου τοῦ ἐκτός, χωρίῳ τῷ περιεχομένῳ ὑπὸ τῶν περιφερειῶν τῶν ἁμικυκλίων, ὅπερ σελήνιον καλείσθω, ἴσος ἐστίν.



Ἔστω ἁμικύκλιον, οὗ διάμετρος ἁ **ΑΒ**, καὶ ἀπὸ τῶν περάτων τᾶς διαμέτρου τῶν **Α**, **Β** δύο τμάματα ἴσα ἀλλάλοις λελάφθω τὰ **ΑΓ**, **ΒΔ**, γεγράφθω δὲ ἀπὸ τῶν τμαμάτων δύο ἁμικύκλια ἐντός, καὶ ἀπὸ τοῦ λοιποῦ τμάματος τοῦ **ΓΔ** γεγράφθω ἁμικύκλιον ἐκτός, διὰ κέντρου δὲ τοῦ ἁμικυκλίου τοῦ **Ε** διαμέτρῳ τᾷ **ΑΒ** ἄχθω ποτ᾽ ὀρθὰς εὐθεῖα ἁ **ΕΖ** καὶ ἐκβεβλήσθω ἐπὶ τὸ **Η** σαμεῖον· φαμὶ δή, ὁ κύκλος, οὗ διάμετρος ἁ **ΖΗ**, χωρίῳ τῷ περιεχομένῳ ὑπὸ τῶν περιφερειῶν τῶν ἁμικυκλίων, ὅπερ σελήνιον καλείσθω, ἴσος ἐστίν.

Ἐπεὶ γὰρ εὐθεῖα γραμμὰ ἁ **ΔΓ** δίχα τέτμαται κατὰ τὸ **Ε** σαμεῖον, ποτίκειται δὲ αὐτᾷ εὐθεῖα ἐπ᾽ εὐθείας ἁ **ΓΑ**, τὸ ἀπὸ τᾶς **ΔΑ** καὶ τὸ ἀπὸ τᾶς ποτικειμένας τᾶς **ΓΑ** τὰ συναμφότερα τετράγωνα διπλασιόνά ἐντι τοῦ τε ἀπὸ τᾶς ἁμισείας τᾶς **ΔΕ** καὶ τοῦ ἀπὸ τᾶς **ΕΑ** τετραγώνου.

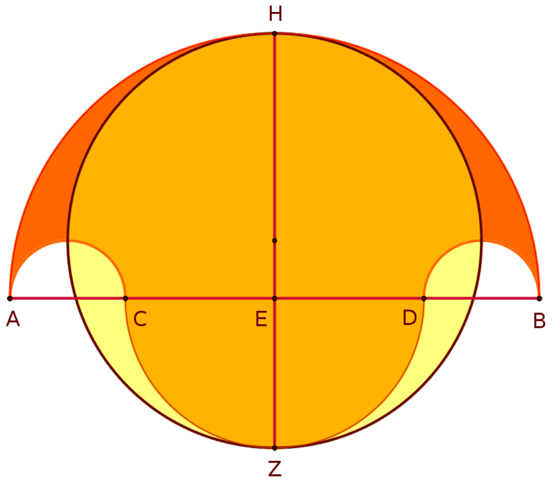
Ἔστι δὲ ἁ **ΖΗ** τᾷ **ΔΑ** ἴσα ἔστιν ἄρα καὶ τὰ ἀπὸ τῶν **ΖΗ**, **ΓΑ** διπλασίονα τοῦ τε ἀπὸ τᾶς **ΔΕ** καὶ τοῦ ἀπὸ τᾶς **ΕΑ**.

Καὶ ἐπεὶ ἁ **ΑΒ** τᾶς **ΑΕ** διπλασίων ἐστὶ καὶ ἁ **ΓΔ** τᾶς **ΔΕ**, ἐσσοῦνται καὶ τὰ ἀπὸ τῶν **ΑΒ**, **ΓΔ** τοῖς ἀπὸ τῶν **ΔΕ**, **ΕΑ** τετραπλασίονα, τουτέστι τοῖς ἀπὸ τῶν **ΖΗ**, **ΓΑ** διπλασίονα· κύκλοι ἄρα, ὧν διάμετροι αἱ **ΑΒ**, **ΔΓ** εὐθεῖαι, κύκλων, ὧν διάμετροι αἱ **ΖΗ**, **ΓΑ**, διπλασιόνές ἐντι· ἁμικύκλια ἄρα, ὧν διάμετροι αἱ **ΑΒ**, **ΔΓ** εὐθεῖαι, κύκλοις, ὧν διάμετροι αἱ **ΖΗ**, **ΓΑ**, ἴσα ἐστίν· κοινὸν ἀφαιρήσθω κύκλος, οὗ διάμετρος ἁ **ΑΓ**, τουτέστι δύο ἁμικύκλια, ὧν διάμετροι αἱ **ΑΓ**, **ΔΒ**· λοιπὸν ἄρα χωρίον τὸ ὑπὸ τῶν περιφερειῶν τῶν ἁμικυκλίων περιεχόμενον, ὅπερ σελήνιον καλεῖται, κύκλῳ, οὗ διάμετρος ἁ **ΖΗ**, ἴσον ἐστίν· δῆλον οὖν τὸ προτεθέν.

14.

If from the end points of the diameter of a semicircle equal segments are measured, and on these segments as diameters inner semicircles are described, and from the rest segment of the diameter an outer semicircle is described, then the area of the circle with diameter the sum of the radii of the central semicircles is equal to the area of the figure bounded by the circumferences of the three semicircles, which shall be called selinio.

Let **AB** be the diameter of a semicircle, and from the end points **A**, **B** of the diameter let **AC**, **BD** be two equal segments measured. On these segments as diameters describe two inner semicircles and on the rest segment **CD** describe an outer semicircle. Let the perpendicular to **AB** through **E**, the centre of the first semicircle, meet the opposite semicircles at **H**, **Z** respectively, I say that the area of the circle with diameter **ZH** is equal to the area of the figure bounded by the circumferences of the three semicircles, which shall be called selinio.



Since **DC** is bisected at **E** and produced to **A** then be Euclid 2.10, will hold

$$DA^2 + AC^2 = 2\cdot(ED^2 + EA^2)$$

but also holds that

$$HZ = EA + ED = DA$$

therefore

$$AB^2 + CD^2 = 4\cdot(ED^2 + EA^2) = 2\cdot(DA^2 + AC^2) = 2\cdot(HZ^2 + AC^2)$$

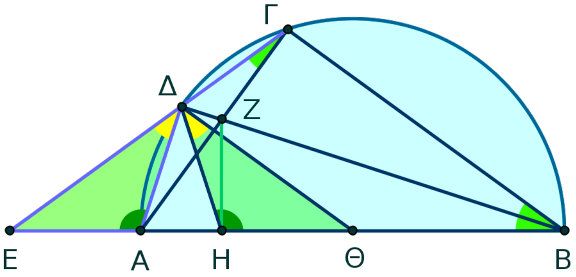
And since circles are to one another as the squares of their diameters and denoting by (C_{XY}) the area of a circle with diameter **XY** and by (S_{XY}) the area of a semicircle with diameter **XY** then we take from above

$$(C_{AB}) + (C_{CD}) = 2\cdot((C_{HZ}) + (C_{AC})) \rightarrow$$
$$(S_{AB}) + (S_{CD}) = (C_{HZ}) + (C_{AC}) \rightarrow$$
$$(S_{AB}) + (S_{CD}) - (C_{AC}) = (C_{HZ}) \rightarrow$$
$$(S_{AB}) + (S_{CD}) - (S_{AC}) - (S_{DB}) = (C_{HZ}) \rightarrow$$
$$(E_{selinio}) = (C_{HZ})$$

Therefore the claim has been proved.

ιε΄.

Ἔστω ἁμικύκλιον τὸ ΑΒ καὶ ἁ ΑΓ πλευρὰ τοῦ ἐγγεγραμμένου ἰσοπλεύρου τε καὶ ἰσογωνίου πενταγώνου, τετμάσθω δὲ περιφέρεια ἁ ΑΓ δίχα κατὰ τὸ Δ, ἐπιζευχθεῖσα δὲ ἁ ΓΔ ἐκβεβλήσθω ἐπὶ τὸ Ε, καὶ ἀπὸ τοῦ Δ σαμείου διάχθω ἁ ΔΒ τεμνέουσα πλευρὰν τὰν ΑΓ κατὰ τὸ Ζ, καὶ ἀπὸ τοῦ Ζ ἄχθω τᾷ ΑΒ ποτ᾽ ὀρθὰς ἁ ΖΗ· φαμὶ δή, εὐθεῖα ἁ ΕΗ τᾷ ἐκ τοῦ κέντρου τοῦ κύκλου ἴσα ἐστίν.



Ἐπεζεύχθω γὰρ ἁ **ΓΒ**, καὶ ἔστω κέντρον τοῦ κύκλου τὸ **Θ** σαμεῖον, καὶ ἄχθωσαν αἱ **ΘΔ**, **ΔΗ**, **ΑΔ** εὐθεῖαι.

Ἐπεὶ οὖν γωνία ἁ ὑπὸ **ΑΒΓ** δύο πέμπτα ὀρθᾶς ἐστιν, γωνία ἁ ὑπὸ **ΓΒΔ**, τουτέστι ἁ ὑπὸ **ΔΒΑ**, ἓν πεμπταμόριον ὀρθᾶς ἐστιν· γωνία ἄρα ἁ ὑπὸ **ΔΘΑ** δύο πέμπτα ὀρθᾶς ἐστιν.

Καὶ ἐπεὶ ἐν δυσὶ τριγώνοις τοῖς **ΓΒΖ**, **ΗΒΖ** δύο γωνίαι αἱ ποτὶ τῷ **Β** ἴσαι ἀλλάλαις ἐντί, ὀρθαὶ δὲ αἱ ποτὶ τὰ **Η**, **Γ** σαμεῖα, κοινὰ δὲ πλευρὰ ἁ **ΒΖ**, ἐσσεῖται ἄρα καὶ βάσις ἁ **ΒΓ** βάσει τᾷ **ΒΗ** ἴσα.

Πάλιν ἐπεὶ ἐν δυσὶ τριγώνοις τοῖς **ΓΒΔ**, **ΗΒΔ** δύο πλευραὶ αἱ **ΓΒ**, **ΒΗ** ἴσαι ἀλλάλαις ἐντί, γωνίαι δὲ αἱ ποτὶ τῷ **Β** ἴσαι, κοινὰ δὲ πλευρὰ ἁ **ΒΔ**, ἐσσεῖται ἄρα γωνία ἁ ὑπὸ **ΒΓΔ** γωνίᾳ τᾷ ὑπὸ **ΒΗΔ**, τουτέστιν ἐπιπέμπτῳ ὀρθᾶς, ἴσα· ἔστι δὲ ἑκατέρα τῶν ὑπὸ **ΒΓΔ**, **ΒΗΔ** γωνιῶν γωνίᾳ τᾷ ἐκτὸς τοῦ ἐν τῷ κύκλῳ τετραπλεύρου τοῦ **ΒΑΔΓ**, τουτέστι τᾷ **ΔΑΕ**, ἴσα· γωνία ἄρα ἁ ὑπὸ **ΔΑΒ** γωνίᾳ τᾷ ὑπὸ **ΔΗΑ** ἔστιν ἴσα, καὶ πλευρὰ ἁ **ΔΑ** τᾷ **ΔΗ**.

Καὶ ἐπεὶ γωνία ἁ ὑπὸ **ΔΘΗ βε΄** ὀρθᾶς ἐστι καὶ ἁ ὑπὸ **ΔΗΘ** ἐπίπεμπτος ὀρθᾶς, γωνία ἄρα ἁ ὑπὸ **ΘΔΗ βε΄** ὀρθᾶς ἐστιν πλευρὰ ἄρα ἁ **ΔΗ** πλευρᾷ τᾷ **ΗΘ** ἐστὶν ἴσα.

Πάλιν, ἐπεὶ γωνία ἁ ὑπὸ **ΑΔΕ** τοῦ ἐν τῷ κύκλῳ τετραπλεύρου τοῦ **ΑΔΓΒ** ἐκτός ἐστιν, ἐσσεῖται ἄρα γωνία ἁ ὑπὸ **ΑΔΕ** γωνίᾳ τᾷ ὑπὸ **ΑΒΓ** ἴσα· ἔστι δὲ γωνία ἁ ὑπὸ **ΑΒΓ βγ΄** ὀρθᾶς· γωνία ἄρα ἁ ὑπὸ **ΑΔΕ** γωνίᾳ τᾷ ὑπὸ **ΗΔΘ** ἐστὶν ἴσα.

Καὶ ἐπεὶ ἐν δυσὶ τριγώνοις τοῖς **ΕΔΑ**, **ΘΔΗ** δύο γωνίαι αἱ ὑπὸ **ΕΔΑ**, **ΔΑΕ** δυσὶ ταῖς ὑπὸ **ΘΔΗ**, **ΔΗΘ** ἑκατέρα ἑκατέρᾳ ἴσαι ἐντί, βάσις δὲ ἁ **ΔΑ** βάσει τᾷ **ΔΗ** ἴσα, πλευρὰ ἄρα ἁ **ΕΑ** πλευρᾷ τᾷ **ΘΗ** ἴσα ἐστίν.

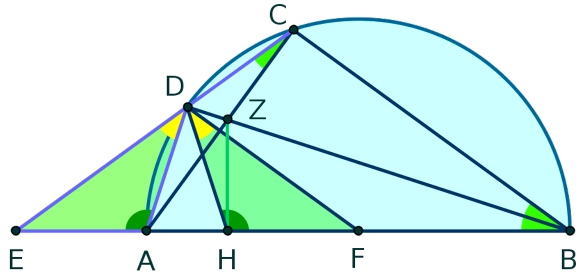
Κοινὰ ποτικείσθω ἁ **ΑΗ**· εὐθεῖα ἄρα ἁ **ΕΗ** εὐθείᾳ τᾷ **ΑΘ**, τουτέστι τᾷ ἐκ τοῦ κέντρου τοῦ κύκλου, ἴσα ἐστίν· δέδεικται οὖν τὸ προτεθέν.

Πόρισμα : Ἐκ τούτου δὴ φανερὸν ὅτι εὐθεῖα ἁ ΔΕ τᾷ ἐκ τοῦ κέντρου τοῦ κύκλου ἐστὶν ἴσα. Ἐπεὶ γὰρ γωνία ἁ ὑπὸ ΔΑΕ γωνίᾳ τᾷ ὑπὸ ΔΗΘ ἴσα ἐστίν, ἐσσεῖται καὶ πλευρὰ ἁ ΔΘ πλευρᾷ τᾷ ΔΕ, τουτέστι τᾷ ΑΘ, ἴσα.

Πόρισμα : Καὶ ἔτι δῆλον ὅτι εὐθεῖα ἁ ΕΓ ἄκρον καὶ μέσον τέτμαται κατὰ τὸ Δ σαμεῖον· τμᾶμα δὲ τὸ ΔΕ τὸ μεῖζόν ἐστιν, ἐπεὶ ἁ ΕΔ πλευρὰ τοῦ ἑξαγώνου, ἁ δὲ ΔΓ πλευρὰ τοῦ δεκαγώνου τῶν ἐν τῷ κύκλῳ ἐγγραφομένων.

15.

Let **AB** be the diameter of a semicircle and **AC** a side of an inscribed regular pentagon. Bisect arc **AC** at point **D**, join **CD** and produce it to meet **BA** in **E**. Join **DB** meeting **AC** in **Z**, and from **Z** draw **ZH** perpendicular to **AB**. I say that **EH** is equal to the radius of the circle.



Join **CB**, and let point **F** be the centre of the circle, and join line segments **FD**, **DH**, **AD**. Then

$$\angle ABC = \frac{2^{L}}{5} \rightarrow$$
$$\angle CBD = \angle DBA = \frac{1^{L}}{5} \rightarrow$$
$$\angle DFA = \frac{2^{L}}{5}$$

For triangles **CBZ**, **HBZ** we have

$$\left.\begin{array}{l} \angle H = \angle C = 1^{L} \\ \angle CBZ = \angle ZBH = \dfrac{1^{L}}{5} \\ BZ \text{ common} \end{array}\right\} \rightarrow BC = BH$$

Then the triangles **CBD**, **HBD** will be equal because

$$\left.\begin{array}{l} CB = BH \\ \angle CBD = \angle DBH \\ BD \text{ common} \end{array}\right\} \rightarrow \angle BHD = \angle BCD = \frac{6^{L}}{5} = \angle DAE \rightarrow \angle DAB = \angle DHA \rightarrow DA = AH$$

Further since angle **ADE** is exterior of the inscribed **ADCB** implies that $\angle ADE = \angle ABC = 2^{L}/5$. Therefore for the triangles **EDA**, **FDH** will be

$$\left.\begin{array}{l} \angle ADE = \angle HDF = \dfrac{2^{L}}{5} \\ DA = DH \\ \angle DAE = \angle DHF = \dfrac{6^{L}}{5} \end{array}\right\} \rightarrow \triangle EDA = FDH\triangle \rightarrow EA = HF \rightarrow EH = AF$$

Therefore the claim has been proved.

Corollary: It is obvious from this that line segment **DE** is equal to the radius of the circle. This is because $\angle DAE = \angle DHF$ implies that $DF = DE = AF$.

Corollary: Also it is obvious that line segment **EC** is divided at **D** in extreme and mean ratio. If the greater side **DE** is a side of a regular hexagon inscribed to a circle then **DC** is a side of a regular decagon inscribed in the same circle. (Euclid 13.9)

Βιβλιογραφία

- J.L. Heiberg, Archimedis, Opera Omnia, Vol. II, B.G. Teubner, Lipsiae
- T.L. Heath, The Works Of Archimedes, Cambridge University Press, 1897
- Charles Mugler, ARCHIMEDE TOME III, PARIS 1971
- E.S. Stamatis Reconstruction of the ancient text (in the sicilian doric dialect) of fifteen theorems of Archimedes which are preserved in the Arabic Language. Bulletin of Hellenic Mathematical Society, New Series, Vol 6II, Issue 2 1965, pages. 265-297

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